Introduction to black hole astrophysics

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Es cosa averiguada que no se sabe nada,
y que todos son ignorantes;
y aun esto no se sabe de cierto,
que, a saberse, ya se supiera algo: sospéchase.

Quevedo

Abstract. Black holes are perhaps the most strange and fascinating objects in the universe. Our understanding of space and time is pushed to its limits by the extreme conditions found in these objects. They can be used as natural laboratories to test the behavior of matter in very strong gravitational fields. Black holes seem to play a key role in the universe, powering a wide variety of phenomena, from X-ray binaries to active galactic nuclei. In these lecture notes the basics of black hole physics and astrophysics are reviewed.

1. Introduction

Strictly speaking, black holes do not exist. Moreover, holes, of any kind, do not exist. You can talk about holes of course. For instance you can say: "there is a hole in the wall". You can give many details of the hole: it is big, it is round shaped, light comes in through it. Even, perhaps, the hole could be such that you can go through to the outside. But I am sure that you do not think that there is a thing made out of nothingness in the wall. No. To talk about the hole is an indirect way of talking about the wall. What really exists is the wall. The wall is made out of bricks, atoms, protons and leptons, whatever. To say that there is a hole in the wall is just to say that the wall has certain topology, a topology such that not every closed curve on the surface of the wall can be contracted to a single point. The hole is not a thing. The hole is a property of the wall.

Let us come back to black holes. What are we talking about when we talk about black holes?. *Space-time*. What is space-time?.

Space-time is the ontological sum of all events of all things.

A thing is an individual endowed with physical properties. An event is a change in the properties of a thing. An ontological sum is an aggregation of things or physical properties, i.e. a physical entity or an emergent property. An ontological sum should not be confused with a set, which is a mathematical construct and has only mathematical (i.e. fictional) properties.

Everything that has happened, everything that happens, everything that will happen, is just an element, a "point", of space-time. Space-time is not a thing, it is just the relational property of all things[1].

As it happens with every physical property, we can represent space-time with some mathematical structure, in order to describe it. We shall adopt the following mathematical structure for space-time:

Space-time can be represented by a C^{∞}-differentiable, 4-dimensional, real manifold.

A real 4-D manifold is a set that can be covered completely by subsets whose elements are in a one-to-one correspondence with subsets of $\Re^4$. Each element of the manifold represents an event. We adopt 4 dimensions because it seems enough to give 4 real numbers to localize an event. For instance, a lightning has beaten the top of the building, located in the 38th Av., between streets 20 and 21, at 25 m above the see level, La Plata city, at 4:35 am, local time, March 2nd, 2009 (this is my home at the time of writing). We see now why we choose a manifold to represent space-time: we can always provide a set of 4 real numbers for every event, and this can be done independently of the intrinsic geometry of the manifold. If there is more than a single characterization of an event, we can always find a transformation law between the different coordinate systems. This is a basic property of manifolds.

Now, if we want to calculate distances between two events, we need more structure on our manifold: we need a geometric structure. We can get this introducing a metric tensor that allows to calculate distances. For instance, consider an Euclidean metric tensor $\delta_{\mu\nu}$ (indices run from 0 to 3):

$$\delta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

Then, adopting the Einstein convention of sum, we have that the distance ds between two arbitrarily close events is:

$$ds^2 = \delta_{\mu\nu} dx^{\mu} dx^{\nu} = (dx^0)^2 + (dx^1)^2 + (dx^3)^2 + (dx^3)^2. \tag{2}$$

Restricted to 3 coordinates, this is the way distances have been calculated since Pythagoras. The world, however, seems to be a little more complicated. After the introduction of the Special Theory of Relativity by Einstein (1905), the German mathematician Hermann Minkowski introduced the following pseudo-Euclidean metric which is consistent with Einstein's theory (Minkowski 1907, 1909):

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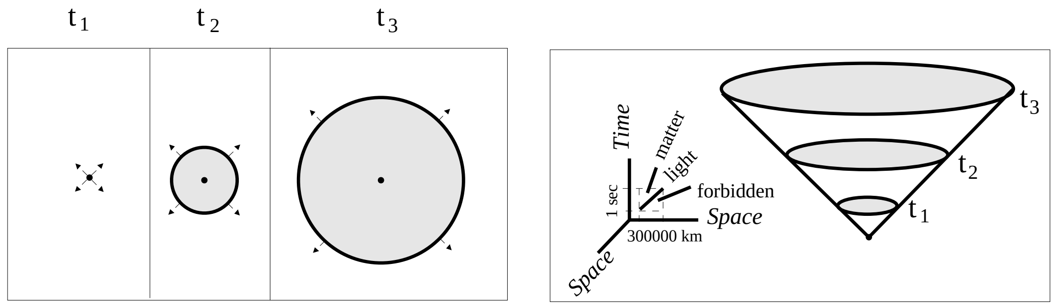
[1] For more details on this view see Perez-Bergliaffa et al. (1998).



(a) Spatial representation (b) The light cone

Figure 1. Light cone. From J-P. Luminet (1998).

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = (dx^0)^2 - (dx^1)^2 - (dx^3)^2 - (dx^3)^2. \tag{3}$$

The Minkowski metric tensor $\eta_{\mu\nu}$ has rank 2 and trace -2. We call the coordinates with the same sign *spatial* (adopting the convention $x^1 = x$, $x^2 = y$, and $x^3 = z$) and the coordinate $x^0 = ct$ is called *temporal coordinate*. The constant c is introduced to make uniform the units. There is an important fact respect to Eq. (3): contrary to what was thought by Kant and others, it is not a necessary statement. Things might have been different. We can easily imagine possible worlds with other metrics. This means that the metric tensor has empirical information about the real universe.

Once we have introduced a metric tensor we can separate space-time at each point in three regions according to $ds^2 < 0$ (space-like region), $ds^2 = 0$ (light-like or null region), and $ds^2 > 0$ (time-like region). Particles that go through the origin can only reach time-like regions. The null surface $ds^2 = 0$ can be inhabited only by particles moving at the speed of light, like photons. Points in the space-like region cannot be reached by material objects from the origin of the *light cone* that can be formed at any space-time point.

The introduction of the metric allows to define the future and the past of a given event. Once this is done, all events can be classified by the relation "earlier than" or "later than". The selection of "present" event - or "now" - is entirely conventional. To be present is not an intrinsic property of any event. Rather, it is a secondary, relational property that requires interaction with a conscious being. The extinction of the dinosaurs will always be earlier than the beginning of World War II. But the latter was present only to some human beings at some physical state. The present is a property like a scent or a color. It emerges from the interaction of self-conscious individuals with changing things and has not existence independently of them (for more about this, see Grünbaum 1973, Chapter X).

Let us consider the unitary vector $T^\nu = (1,\ 0,\ 0,\ 0)$, then a vector x^μ points to the future if $\eta_{\mu\nu} x^\mu T^\nu > 0$. In the similar way, the vector points toward the past if $\eta_{\mu\nu} x^\mu T^\nu < 0$. A light cone is shown in Figure 1.

We define the *proper time* (τ) of a physical system as the time of a co-moving system, i.e. $dx = dy = dz = 0$, and hence:

$$d\tau^2 = \frac{1}{c^2} ds^2. \tag{4}$$

Since the interval is an invariant (i.e. it has the same value in all coordinate systems), it is easy to show that:

$$d\tau = \frac{dt}{\gamma}, \tag{5}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \tag{6}$$

is the Lorentz factor of the system.

A basic characteristic of Minskowski space-time is that it is "flat": all light cones point in the same direction, i.e. the local direction of the future does not depend on the coefficients of the metric since these are constants. More general space-times are possible. If we want to describe gravity in the framework of space-time, we have to introduce a pseudo-Riemannian space-time, whose metric can be flexible, i.e. a function of the material properties (mass-energy and momentum) of the physical systems that produce the events of space-time.

Tetrads: othogonal unit vector fields

Let us consider a scalar product

$$\vec{v} \bullet \vec{w} = (v^\mu \widehat{e}_\mu) \bullet (w^\nu \widehat{e}_\nu) = (\widehat{e}_\mu \bullet \widehat{e}_\nu) v^\mu w^\nu = g_{\mu\nu} v^\mu w^\nu,$$

where

$$\widehat{e}_\mu = \lim_{\delta x^\mu \to 0} \frac{\delta \vec{s}}{\delta x^\mu},$$

and we have defined

$$\widehat{e}_\mu(x) \bullet \widehat{e}_\nu(x) = g_{\mu\nu}(x).$$

Similarly,

$$\widehat{e}^\mu(x) \bullet \widehat{e}^\nu(x) = g^{\mu\nu}(x).$$

We call $\widehat{e}_\mu$ a *coordinate basis vector* or a *tetrad*. $\delta\vec{s}$ is an infinitesimal displacement vector between a point P on the manifold (see Fig. 2) and a nearby point Q whose coordinate separation is δx^μ along the x^μ coordinate curve. $\widehat{e}_\mu$ is the tangent vector to the x^μ curve at P. We can write:

$$d\vec{s} = \widehat{e}_\mu dx^\mu$$

and then:

$$ds^2 = d\vec{s} \bullet d\vec{s} = (dx^\mu \widehat{e}_\mu) \bullet (dx^\nu \widehat{e}_\nu) = (\widehat{e}_\mu \bullet \widehat{e}_\nu) dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu.$$

At a given point P the manifold is flat, so:

$$g_{\mu\nu}(P) = \eta_{\mu\nu}.$$

A manifold with such a property is called *pseudo-Riemannian*. If $g_{\mu\nu}(P) = \delta_{\mu\nu}$ the manifold is called strictly *Riemannian*.

The basis is called *orthonormal* when $\widehat{e}^{\mu} \bullet \widehat{e}_{\nu} = \eta^{\mu}_{\nu}$ at any given point P. Notice that since the tetrads are 4-dimensional we can write:

$$e_{\mu a}(x) e^{a}_{\nu}(x) = g_{\mu\nu}(x),$$

and

$$e_{\mu a}(P) e^{a}_{\nu}(P) = \eta_{\mu\nu}.$$

The tetrads can vary along a given world-line, but always satisfying:

$$e_{\mu a}(\tau) e^{a}_{\nu}(\tau) = \eta_{\mu\nu}.$$

We can also express the scalar product $\vec{v} \bullet \vec{w}$ in the following ways:

$$\vec{v} \bullet \vec{w} = (v_{\mu}\widehat{e}^{\mu}) \bullet (w_{\nu}\widehat{e}^{\nu}) = (\widehat{e}^{\mu} \bullet \widehat{e}^{\nu}) v_{\mu} w_{\nu} = g^{\mu\nu} v_{\mu} w_{\nu},$$

$$\vec{v} \bullet \vec{w} = (v^{\mu}\widehat{e}_{\mu}) \bullet (w_{\nu}\widehat{e}^{\nu}) = (\widehat{e}_{\mu} \bullet \widehat{e}^{\nu}) v^{\mu} w_{\nu} = v^{\mu} w_{\nu} \delta^{\nu}_{\mu} = v^{\mu} w_{\mu},$$

and

$$\vec{v} \bullet \vec{w} = (v_{\mu}\widehat{e}^{\mu}) \bullet (w^{\nu}\widehat{e}_{\nu}) = (\widehat{e}^{\mu} \bullet \widehat{e}_{\nu}) v_{\mu} w^{\nu} = \delta^{\mu}_{\nu} v_{\mu} w^{\nu} = v_{\mu} w^{\mu}.$$

By comparing these expressions for the scalar product of two vectors, we see that

$$g_{\mu\nu} w^{\nu} = w_{\mu},$$

so the quantities $g_{\mu\nu}$ can be used to lower and index. Similarly,

$$g^{\mu\nu} w_{\nu} = w^{\mu}.$$

We also have that

$$g_{\mu\nu} w^{\nu} g^{\mu\nu} w_{\nu} = g_{\mu\nu} g^{\mu\nu} w^{\nu} w_{\nu} = w_{\mu} w^{\mu}.$$

And from here it follows:

$$g^{\mu\nu} g_{\mu\sigma} = \delta^{\nu}_{\sigma}.$$

The tensor field $g_{\mu\nu}(x)$ is called the *metric tensor* of the manifold. Alternative, the metric of the manifold can be specified by the tetrads $e^{a}_{\mu}(x)$.

2. Gravitation

The key to relate space-time to gravitation is the *equivalence principle* introduced by Einstein (1907):

> At every space-time point in an arbitrary gravitational field it is possible to choose a locally inertial coordinate system such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in absence of gravitation (fromulation by Weinberg 1972).

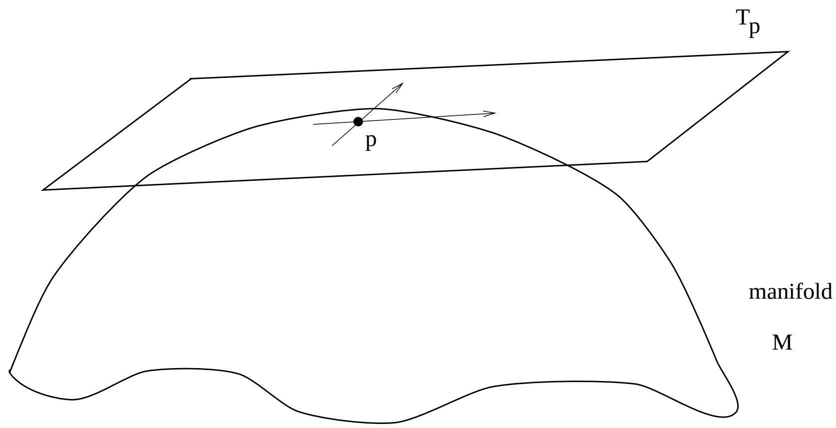


Figure 2. Tangent flat space at a point P of a curved manifold. From Carroll (2003).

This is equivalent to state that at every point P of the manifold that represents space-time there is a flat tangent surface. Einstein called the idea that gravitation vanishes in free-falling systems "the happiest thought of my life" (Pais 1982).

In order to introduce gravitation in a general space-time, we define a metric tensor $g_{\mu\nu}$, such that its components can be related to those of a locally Minkowski space-time defined by $ds^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta$ through a general transformation:

$$ds^2 = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu. \tag{7}$$

In the absence of gravity we can always find a global coordinate system (ξ^α) for which the metric can take the form given by Eq. (3) everywhere. With gravity, on the contrary, such a coordinate system can represent space-time only in an infinitesimal neighborhood of a given point. This situation is represented in Fig 2, where the tangent flat space to a point P of the manifold is shown. The curvature of space-time means that it is not possible to find coordinates in which $g_{\mu\nu} = \eta_{\mu\nu}$ at *all points of the manifold*. However, it is always possible to represent the event (point) P in a system such that $g_{\mu\nu}(P) = \eta_{\mu\nu}$ and $(\partial g_{\mu\nu}/\partial x^\sigma)_P = 0$.

To find the equation of motion of a free particle (i.e. only subject to gravity) in a general space-time of metric $g_{\mu\nu}$ let us consider a freely falling coordinate system ξ^α. In such a system:

$$\frac{d^2\xi^\alpha}{ds^2} = 0, \tag{8}$$

where $ds^2 = (cd\tau)^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta$. Let us consider now any other coordinate system x^μ. Then,

$$\frac{d}{ds}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{ds}\right) = 0 \tag{9}$$

$$\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{ds^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0. \tag{10}$$

Multiplying at both sides by $\partial x^{\lambda}/\partial\xi^{\alpha}$ and using:

$$\frac{\partial\xi^{\alpha}}{\partial x^{\mu}}\frac{\partial x^{\lambda}}{\partial\xi^{\alpha}}=\delta^{\lambda}_{\mu}, \tag{11}$$

we get

$$\frac{d^2x^{\lambda}}{ds^2}+\Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}=0, \tag{12}$$

where $\Gamma^{\lambda}_{\mu\nu}$ is the affine connection of the manifold:

$$\Gamma^{\lambda}_{\mu\nu}\equiv\frac{\partial x^{\lambda}}{\partial\xi^{\alpha}}\frac{\partial^2\xi^{\alpha}}{\partial x^{\mu}\partial x^{\nu}}. \tag{13}$$

The affine connection can be expressed in terms of derivatives of the metric tensor (see, e.g., Weinberg 1972):

$$\Gamma^{\lambda}_{\mu\nu}=\frac{1}{2}g^{\lambda\alpha}(\partial_{\mu}g_{\nu\alpha}+\partial_{\nu}g_{\mu\alpha}-\partial_{\alpha}g_{\mu\nu}). \tag{14}$$

Here we use the convention: $\partial_{\nu}f=\partial f/\partial x^{\nu}$ and $g^{\mu\alpha}g_{\alpha\nu}=\delta^{\mu}_{\nu}$. Notice that under a coordinate transformation from x^{μ} to $x^{'\mu}$ the affine connection is not transformed as a tensor, despite that the metric $g_{\mu\nu}$ is a tensor of second rank.

The coefficients $\Gamma^{\lambda}_{\mu\nu}$ are said to define a *connection* on the manifold. ăWhat are connected are the tangent spaces at different points of the manifold. It is then possible to compare a vector in the tangent space at point P with the vector parallel to it at another point Q. There is some degree of freedom in the specification of the affine connection, so we demand symmetry in the last two indices:

$$\Gamma^{\lambda}_{\mu\nu}=\Gamma^{\lambda}_{\nu\mu}$$

or

$$\Gamma^{\lambda}_{[\mu\nu]}=0.$$

In general space-times this requirement is not necessary, and a tensor can be introduced such that:

$$T^{\lambda}_{\mu\nu}=\Gamma^{\lambda}_{[\mu\nu]}. \tag{15}$$

This tensor represents the *torsion* of space-time. In general relativity space-time is always considered as torsionless, but in the so-called *teleparallel* theory of gravity (e.g. Arcos and Pereira 2004) torsion represents the gravitational field instead of curvature, which is nil.

In a pseudo-Riemannian space-time the usual partial derivative is not a meaningful quantity since we can give it different values through different choices of coordinates. This can be seen in the way the derivative transforms under a coordinate change:

$$A'^{\mu}_{,\nu}=\frac{\partial}{\partial x'^{\nu}}\frac{\partial x'^{\mu}}{\partial x^{\mu}}\,A^{\mu}=\frac{\partial x'^{\mu}}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial x'^{\nu}}\,A^{\mu}_{,\nu}+\frac{\partial^2 x'^{\mu}}{\partial x^{\mu}\partial x^{\nu}}\frac{\partial x^{\nu}}{\partial x'^{\nu}}\,A^{\mu}. \tag{16}$$

We can define a covariant differentiation through the condition of parallel transport:

$$A_{\mu};_{\nu}=\frac{\partial A_{\mu}}{\partial x^{\nu}}-\Gamma^{\lambda}_{\mu\nu}A_{\lambda}. \tag{17}$$

A useful, alternative notation, is:

$$\nabla_\nu A_\mu = \frac{\partial A_\mu}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu} A_\lambda. \tag{18}$$

A covariant derivative of a vector field is a rank 2 tensor of type (1, 1). A covariant divergence of a vector field yields a scalar field:

$$\nabla_\mu A^\mu = \partial_\mu A^\mu(x) - \Gamma^\mu_{\alpha\mu} A^\alpha(x) = \phi(x). \tag{19}$$

A tangent vector satisfies $V^\nu V_\nu;_\mu = 0$. If there is a vector ζ^μ pointing in the direction of a symmetry of space-time, then it can be shown (e.g. Weinberg 1972):

$$\zeta_\mu;_\nu + \zeta_\nu;_\mu = 0, \tag{20}$$

or

$$\nabla_\nu \zeta_\mu + \nabla_\mu \zeta_\nu = 0. \tag{21}$$

This equation is called Killing's equation. A vector field ζ^μ satisfying such a relation is called a Killing vector.

If there is a curve γ on the manifold, such that its tangent vector is $u^\alpha = dx^\alpha/d\lambda$ and a vector field A^α is defined in a neighborhood of γ, we can define a derivative of A^α along γ as:

$$\ell_\mu A^\alpha = A^\alpha_{,\beta} u^\beta - u^\alpha_{,\beta} A^\beta = A^\alpha_{;\beta} u^\beta - u^\alpha_{;\beta} A^\beta. \tag{22}$$

This derivative is a tensor, and it is usually called *Lie derivative*. It can be defined for tensor of any type. A Killing vector field is such that:

$$\ell_\zeta g_{\mu\nu} = 0. \tag{23}$$

From Eq. (12) we can recover the classical Newtonian equations if:

$$\Gamma^0_{i,j} = 0, \;\; \Gamma^i_{0,j} = 0, \;\; \Gamma^i_{0,0} = \frac{\partial \Phi}{\partial x^i},$$

where $i, \; j = 1, 2, 3$ and Φ is the Newtonian gravitational potential. Then:

$$x^0 = ct = c\tau,$$

$$\frac{d^2 x^i}{d\tau^2} = -\frac{\partial \Phi}{\partial x^i}.$$

We see, then, that the metric represents the gravitational potential and the affine connection the gravitational field.

The presence of gravity is indicated by the curvature of space-time. The Riemann tensor, or curvature tensor, provides a measure of this curvature:

$$R^\sigma_{\mu\nu\lambda} = \Gamma^\sigma_{\mu\lambda,\nu} - \Gamma^\sigma_{\mu\nu,\lambda} + \Gamma^\sigma_{\alpha\nu}\Gamma^\alpha_{\mu\lambda} - \Gamma^\sigma_{\alpha\lambda}\Gamma^\alpha_{\mu\nu}. \tag{24}$$

The form of the Riemann tensor for an affine-connected manifold can be obtained through a coordinate transformation $x^\mu \to \bar{x}^\mu$ that makes the affine connection to vanish everywhere, i.e.

$$\bar{\Gamma}^\sigma_{\mu\nu}(\bar{x}) = 0, \quad \forall \bar{x}, \; \rho, \; \mu, \; \nu. \tag{25}$$

The coordinate system $\bar{x}^{\mu}$ exists iff:

$$\Gamma^{\sigma}_{\mu\lambda,\nu} - \Gamma^{\sigma}_{\mu\nu,\lambda} + \Gamma^{\sigma}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda} - \Gamma^{\sigma}_{\alpha\lambda}\Gamma^{\alpha}_{\mu\nu} = 0, \tag{26}$$

for the affine connection $\Gamma^{\sigma}_{\mu\nu}(x)$. The right hand side of Eq. (26) is the Riemann tensor: $R^{\sigma}_{\mu\nu\lambda}$. In such a case the metric is flat, since its derivatives are zero. If $R^{\sigma}_{\mu\nu\lambda} > 0$ the metric has a positive curvature.

The Ricci tensor is defined by:

$$R_{\mu\nu} = g^{\lambda\sigma} R_{\lambda\mu\sigma\nu} = R^{\sigma}_{\mu\sigma\nu}. \tag{27}$$

Finally, the Ricci scalar is $R = g^{\mu\nu} R_{\mu\nu}$.

3. Field equations

The key issue to determine the geometric structure of space-time, and hence to specify the effects of gravity, is to find the law that fixes the metric once the source of the gravitational field is given. The source of the gravitational field is the energy-momentum tensor $T_{\mu\nu}$ that represents the physical properties of a material thing. This was Einstein's fundamental intuition: the curvature of space-time at any event is related to the energy-momentum content at that event. For the simple case of a perfect fluid the energy-momentum tensor takes the form:

$$T_{\mu\nu} = (\epsilon + p) u_{\mu} u_{\nu} - p g_{\mu\nu}, \tag{28}$$

where ϵ is the mass-energy density, p is the pressure, and $u^{\mu} = dx^{\mu}/ds$ is the 4-velocity. The field equations were found by Einstein (1915) and independently by Hilbert (1915) on November 25th and 20th, 1915, respectively[2].

We can write Einstein's physical intuition in the following form:

$$K_{\mu\nu} = \kappa T_{\mu\nu}, \tag{29}$$

where $K_{\mu\nu}$ is a rank-2 tensor related to the curvature of space-time and κ is a constant. Since the curvature is expressed by $R_{\mu\nu\sigma\rho}$, $K_{\mu\nu}$ must be constructed from this tensor and the metric tensor $g_{\mu\nu}$. The tensor $K_{\mu\nu}$ has the following properties to satisfy: i) the Newtonian limit suggests that it should contain terms no higher than linear in the second-order of derivatives of the metric tensor (since $\nabla^2\Phi = 4\pi G\rho$); ii) since $T_{\mu\nu}$ is symmetric then $K_{\mu\nu}$ must be symmetric as well. Since $R_{\mu\nu\sigma\rho}$ is already linear in the second-order derivatives of the metric, the most general form of $K_{\mu\nu}$ is:

$$K_{\mu\nu} = aR_{\mu\nu} + bRg_{\mu\nu} + \lambda g_{\mu\nu}, \tag{30}$$

where a, b, and λ are constants.

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[2] Recent scholarship has arrived to the conclusion that Einstein was the first to find the equations and that Hilbert incorporated the final form of the equations in the proof reading process, after Einstein's communication (Corry et al. 1997).

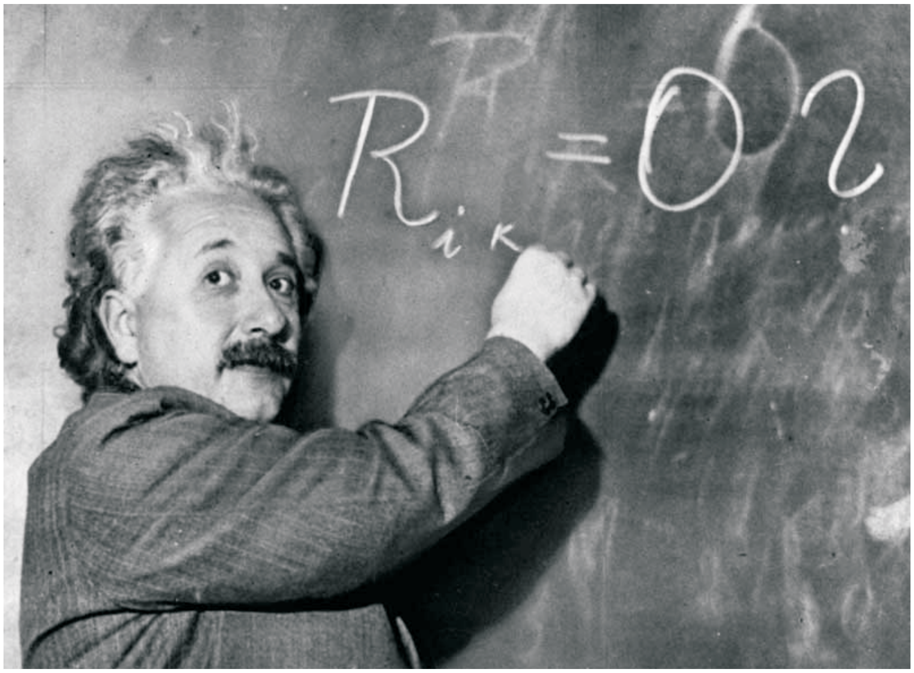

Figure 3. Albert Einstein and the field equations of General Relativity for empty space.

If *every* term in $K_{\mu\nu}$ must be linear in the second-order derivatives of $g_{\mu\nu}$, then $\lambda = 0$. Hence:

$$K_{\mu\nu} = aR_{\mu\nu} + bRg_{\mu\nu}. \tag{31}$$

The conservation of energy-momentum requires: $T^{\mu\nu};\mu = 0$. So,

$$(aR^{\mu\nu} + bRg^{\mu\nu});\mu = 0. \tag{32}$$

Also, it happens that (Bianchi identities):

$$(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu});\mu = 0. \tag{33}$$

From here, we get $b = -a/2$ and $a = 1$. We can then re-write the field equations as:

$$(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) = \kappa T_{\mu\nu}. \tag{34}$$

In order to fix κ, we must compare with the weak-field limit of these equations with the Poisson's equations of Newtonian gravity. This requires that $\kappa = -8\pi G/c^4$.

The Einstein field equations can then be written in the simple form:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -(8\pi G/c^4)T_{\mu\nu}. \tag{35}$$

This is a set of ten non-linear partial differential equations for the metric coefficients. In Newtonian gravity, otherwise, there is only one gravitational field equation. General Relativity involves numerous non-linear differential equations. In this fact lays its complexity, and its richness.

The conservation of mass-energy and momentum can be derived from the field equations:

$$T^{\mu\nu};_{\nu} = 0 \quad \text{or} \quad \nabla_{\nu} T^{\mu\nu} = 0. \tag{36}$$

Contrary to classical electrodynamics, here the field equations entail the energy-momentum conservation and the equations of motion for free particles (i.e. for particles moving in the gravitational field, treated here as a background pseudo-Riemannian space-time).

Let us consider, for example, a distribution of dust (i.e. a pressureless perfect fluid) for which the energy-momentum tensor is:

$$T^{\mu\nu} = \rho u^{\mu} u^{\nu}, \tag{37}$$

with u^{μ} the 4-velocity. Then,

$$T^{\mu\nu};_{\mu} = (\rho u^{\mu} u^{\nu});_{\mu} = (\rho u^{\mu});_{\mu} u^{\nu} + \rho u^{\mu} u^{\nu};_{\mu} = 0. \tag{38}$$

Contracting with u_{ν}:

$$c^2 (\rho u^{\mu});_{\mu} + (\rho u^{\mu}) u_{\nu} u^{\nu};_{\mu} = 0, \tag{39}$$

where we used $u^{\nu} u_{\nu} = c^2$. Since the second term on the left is zero, we have:

$$(\rho u^{\mu});_{\mu} = 0. \tag{40}$$

Replacing in Eq. (38), we obtain:

$$u^{\mu} u^{\nu};_{\nu} = 0, \tag{41}$$

which is the equation of motion for the dust distribution in the gravitational field.

Einstein equations (35) can be cast in the form:

$$R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} R = -(8\pi G/c^4) T^{\mu}_{\nu}. \tag{42}$$

Contracting by setting $\mu = \nu$ we get

$$R = -(16\pi G/c^4) T, \tag{43}$$

where $T = T^{\mu}_{\mu}$. Replacing the curvature scalar in Eqs. (35) we obtain the alternative form:

$$R_{\mu\nu} = -(8\pi G/c^4)(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}). \tag{44}$$

In a region of empty space, $T_{\mu\nu} = 0$ and then

$$R_{\mu\nu} = 0, \tag{45}$$

i.e. the Ricci tensor vanishes. The curvature tensor, which has 20 independent components, does not necessarily vanishes. This means that a gravitational field can exist in empty space only if the dimensionality of space-time is 4 or higher. For space-times with lower dimensionality, the curvature tensor vanishes if $T_{\mu\nu} = 0$. The components of the curvature tensor that are not zero in empty space are contained in the Weyl tensor (see Section 8. below for a definition of the Weyl tensor). Hence, the Weyl tensor describes the curvature of empty space. Absence of curvature (flatness) demands that both a Ricci and Weyl tensors should be zero.

4. The cosmological constant

The set of Einstein equations is not unique: we can add any constant multiple of $g_{\mu\nu}$ to the left member of (35) and still obtain a *consistent* set of equations. It is usual to denote this multiple by Λ, so the field equations can also be written as:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -(8\pi G/c^4) T_{\mu\nu}. \tag{46}$$

Lambda is a new universal constant called, because of historical reasons, the *cosmological constant*. If we consider some kind of "substance" with equation of state given by $p = -\rho c^2$, then its energy-momentum tensor would be:

$$T_{\mu\nu} = -p g_{\mu\nu} = \rho c^2 g_{\mu\nu}. \tag{47}$$

Notice that the energy-momentum tensor of this substance depends only on the space-time metric $g_{\mu\nu}$, hence it describes a property of the "vacuum" itself. We can call ρ the energy density of the vacuum field. Then, we rewrite Eq. (46) as:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -(8\pi G/c^4)(T_{\mu\nu} + T_{\mu\nu}^{\mathrm{vac}}), \tag{48}$$

in such a way that

$$\rho_{\mathrm{vac}} c^2 = \frac{\Lambda c^4}{8\pi G}. \tag{49}$$

There is evidence (e.g. Perlmutter et al. 1999) that the energy density of the vacuum is different from zero. This means that Λ is small, but not zero[3]. The negative pressure seems to be driving a "cosmic acceleration".

There is a simpler interpretation of the repulsive force that produces the accelerate expansion: there is not a dark field. The only field is gravity, represented by $g_{\mu\nu}$. What is different is the law of gravitation: instead of being given by Eqs. (35), it is expressed by Eqs. (46); gravity can be repulsive under some circumstances.

Despite the complexity of Einstein's field equations a large number of exact solutions have been found. They are usually obtained imposing symmetries on the space-time in such a way that the metric coefficients can be found. The first and most general solution to Eqs. (35) was obtained by Karl Schwarszchild in 1916, short before he died in the Eastern Front of World War I. This solution, as we will see, describes a non-rotating black hole of mass M.

5. Relativistic action

Let us consider a mechanical system whose configuration can be uniquely defined by generalized coordinates q^a, $a = 1, 2, ..., n$. The *action* of such a system is:

$$S = \int_{t_1}^{t_2} L(q^a, \dot{q}^a, t) dt, \tag{50}$$

---

[3]The current value is around 10^{-29}g cm^{-3}.

where t is the time. The *Lagrangian* L is defined in terms of the kinetic energy T of the system and the potential energy U:

$$L = T - U = \frac{1}{2} m g_{ab} \dot{q}^a \dot{q}^b - U, \tag{51}$$

where g_{ab} is the metric of the configuration space: $ds^2 = g_{ab} dq^a dq^b$. Hamilton's principle states that for arbitrary variation such as:

$$q^a(t) \to q'^a(t) = q^a(t) + \delta q^a(t), \tag{52}$$

the variation of the action δS vanishes. Assuming that $\delta q^a(t) = 0$ at the end-points t_1 and t_2 of the trajectory, it can be shown that the Lagrangian must satisfy the Euler-Lagrange equations:

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = 0, \quad a = 1, 2, ..., n. \tag{53}$$

These are the equations of motion of the system.

In the case of the action of a set of fields defined on some general four dimensional space-time manifold, we can introduce a *Lagrangian density* of the fields and their derivatives:

$$S = \int_{\mathcal{R}} \mathcal{L}(\Phi^a,\ \partial_\mu \Phi^a,\ \partial_\mu \partial_\nu \Phi^a ...)\ d^4x, \tag{54}$$

where Φ^a is a field on the manifold, $\mathcal{R}$ is a region of the manifold, and $d^4x = dx^0 dx^1 dx^2 dx^3$. The action should be a scalar, then we should use the element of volume in a system x^μ written in the invariant form $\sqrt{-g}\ d^4x$, where $g = \|g^{\mu\nu}\|$ is the determinant of the metric in that coordinate system. The corresponding action is:

$$S = \int_{\mathcal{R}} L \sqrt{-g}\ d^4x, \tag{55}$$

where the *Lagrangian field* L is related with the Lagrangian density by:

$$\mathcal{L} = L\sqrt{-g}. \tag{56}$$

The field equations of Φ^a can be derived demanding that the action (54) is invariant under small variations in the fields:

$$\Phi^a(x) \to \Phi'^a(x) = \Phi^a + \delta\Phi^a(x). \tag{57}$$

No coordinate has been changed here, just the form of the fields in a fixed coordinate system. Assuming, for simplicity that the field is local, higher order derivatives can be neglected. For first order derivatives we have:

$$\partial_\mu \Phi^a \to \partial_\mu \Phi'^a = \partial^a_\mu + \partial_\mu(\delta\Phi^a). \tag{58}$$

Using these variations, we obtain the variation of the action $S \to S + \delta S$, where:

$$\delta S = \int_{\mathcal{R}} \delta\mathcal{L} d^4x = \int_{\mathcal{R}} \left[\frac{\partial \mathcal{L}}{\partial \Phi^a}\delta\Phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)}\delta(\partial_\mu \Phi^a)\right]\ d^4x. \tag{59}$$

After some math (see, e.g., Hobson et al. 2007), we get:

$$\frac{\delta \mathcal{L}}{\delta \Phi^a} = \frac{\partial \mathcal{L}}{\partial \Phi^a} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \right] = 0. \tag{60}$$

These are the Euler-Lagrange equations for the local field theory defined by the action (54).

If the field theory is General Relativity, we need to define a Lagrangian density which is a scalar under general coordinate transformations and which depends on the components of the metric tensor $g_{\mu\nu}$, which represents the dynamical potential of the gravitational field. The simplest scalar that can be constructed from the metric and its derivatives is the Ricci scalar R. The simplest possible action is the so-called *Einstein-Hilbert* action:

$$S_{\rm EH} = \int_{\mathcal{R}} R\sqrt{-g}\; d^4x. \tag{61}$$

The Lagrangian density is $\mathcal{L} = R\sqrt{-g}$. Introducing a variation in the metric

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \tag{62}$$

we can arrive, after significant algebra, to:

$$\delta S_{\rm EH} = \int_{\mathcal{R}} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R)\; \delta g^{\mu\nu} \sqrt{-g}\; d^4x. \tag{63}$$

By demanding that $\delta S_{\rm EH} = 0$ and considering that $\delta g_{\mu\nu}$ is arbitrary, we get:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{64}$$

These are the Einstein field equations in vacuum. The tensor $G_{\mu\nu}$ is called the *Einstein tensor*. This variational approach was used by Hilbert in November 1915 to derive the Einstein equations from simplicity and symmetry arguments.

If there are non-gravitational fields present the action will have and additional component:

$$S = \frac{1}{2\kappa} S_{\rm EH} + S_{\rm M} = \int_{\mathcal{R}} \left(\frac{1}{2\kappa} \mathcal{L}_{\rm EH} + \mathcal{L}_{\rm M} \right)\; d^4x, \tag{65}$$

where $S_{\rm M}$ is the non-gravitational action and $\kappa = -8\pi G/c^4$.

If we vary the action with respect to the inverse metric we get:

$$\frac{1}{2\kappa} \frac{\delta \mathcal{L}_{\rm EH}}{\delta g^{\mu\nu}} + \frac{\delta \mathcal{L}_{\rm M}}{\delta g^{\mu\nu}} = 0. \tag{66}$$

Since $\delta S_{\rm EH} = 0$,

$$\frac{\delta \mathcal{L}_{\rm EH}}{\delta g^{\mu\nu}} = \sqrt{-g} G_{\mu\nu}. \tag{67}$$

Then, if we identify the energy-momentum tensor of the non-gravitational fields in the following way:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_{\rm M}}{\delta g^{\mu\nu}}, \tag{68}$$

we obtain the full Einstein equations:

$$G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}.$$

Math note: invariant volume element

Let us calculate the N-dimensional volume element $d^N V$ in an N-dimensional pseudo-Rimannian manifold. In an orthogonal coordinate system this volume element is:

$$d^N V = \sqrt{|g_{11} g_{22} ... g_{NN}|} dx^1 dx^2 ... dx^N.$$

In such a system the metric tensor is such that its determinant is:

$$\|g_{ab}\| = g_{11} g_{22} ... g_{NN},$$

i.e. the product of the diagonal elements.

Using the notation adopted above for the determinant we can write:

$$d^N V = \sqrt{|g|} dx^1 dx^2 ... dx^N.$$

It is not difficult to show that this result remains valid in an arbitrary coordinate system (see Hobson et al. 2007).

6. The Cauchy problem

The Cauchy problem concerns the solution of a partial differential equation that satisfies certain side conditions which are given on a hypersurface in the domain. It is an extension of the initial value problem. In the case of the Einstein field equations, the hypersurface is given by the condition $x^0/c = t$. If it were possible to obtain from the field equations an expression for $\partial^2 g_{\mu\nu}/\partial(x^0)^2$ everywhere at t, then it would be possible to compute $g_{\mu\nu}$ and $\partial g_{\mu\nu}/\partial x^0$ at a time $t + \delta t$, and repeating the process the metric could be calculated for all x^μ. This is the problem of finding the causal development of a physical system from initial data.

Let us prescribe initial data g_{ab} and $g_{ab,0}$ on S defined by $x^0/c = t$. The dynamical equations are the six equations defined by

$$G^{i,j} = -\frac{8\pi G}{c^4} T^{ij}. \tag{69}$$

When these equations are solved for the 10 second derivatives $\partial^2 g_{\mu\nu}/\partial(x^0)^2$, there appears a fourfold ambiguity, i.e. four derivatives are left indeterminate. In order to fix completely the metric it is necessary to impose four additional conditions. These conditions are usually imposed upon the affine connection:

$$\Gamma^\mu \equiv g^{\alpha\beta} \Gamma^\mu_{\alpha\beta} = 0. \tag{70}$$

The condition $\Gamma^\mu = 0$ implies $\Box^2 x^\mu = 0$, so the coordinates are known as harmonic. With such conditions it can be shown the existence, uniqueness and stability of the solutions. But such a result is in no way general and this is an active field of research. The fall of predictability posits a serious problem for the space-time interior of black holes and for multiply connected space-times, as we will see.

7. The energy-momentum of gravitation

Taking the covariant derivative to both sides of Einstein's equations we get, using Bianchi identities:

$$(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)_{;\mu} = 0, \tag{71}$$

and then $T^{\mu\nu}{}_{;\mu} = 0$. This means the conservation of energy and momentum of matter and non-gravitational fields, but it is not strictly speaking a full conservation law, since the energy-momentum of the gravitational field is not included. Because of the Equivalence Principle, it is always possible to choose a coordinate system where the gravitational field locally vanishes. Hence, its energy is zero. Energy is the more general property of things: the potential of change. This property, however, cannot be associated with a pure gravitational field at any point according to General Relativity. Therefore, it is not possible to associate a tensor with the energy-momentum of the gravitational field. Nonetheless, extended regions with gravitational field have energy-momentum since it is impossible to make the field null in all points of the region just through a coordinate change. We can then define a quasi-tensor for the energy-momentum. Quasi-tensors are objects that under *global* linear transformations behave like tensors.

We can define a quasi-tensor of energy-momentum such that:

$$\Theta^{\mu\nu}{}_{,\nu} = 0. \tag{72}$$

In the absence of gravitational fields it satisfies $\Theta^{\mu\nu} = T^{\mu\nu}$. Hence, we can write:

$$\Theta^{\mu\nu} = \sqrt{-g}\,(T^{\mu\nu} + t^{\mu\nu}) = \Lambda^{\mu\nu\alpha}{}_{,\alpha}\,. \tag{73}$$

An essential property of $t^{\mu\nu}$ is that it is not a tensor, since in the *superpotential* on the right side figures the normal derivative, not the covariant one. Since $t^{\mu\nu}$ can be interpreted as the contribution of gravitation to the quasi-tensor $\Theta^{\mu\nu}$, we can expect that it should be expressed in geometric terms only, i.e. as a function of the affine connection and the metric. Landau & Lifshitz (1962) have found an expression for $t^{\mu\nu}$ that contains only first derivatives and is symmetric:

$$\begin{aligned} t^{\mu\nu} \;=\; & \frac{c^4}{16\pi G}\Big[\left(2\Gamma^{\sigma}_{\rho\eta}\Gamma^{\gamma}_{\sigma\gamma} - \Gamma^{\sigma}_{\rho\gamma}\Gamma^{\gamma}_{\eta\sigma} - \Gamma^{\sigma}_{\rho\sigma}\Gamma^{\gamma}_{\eta\gamma}\right)(g^{\mu\rho}g^{\nu\eta} - g^{\mu\nu}g^{\rho\eta}) + \\ & + g^{\mu\rho}g^{\eta\sigma}\left(\Gamma^{\nu}_{\rho\gamma}\Gamma^{\gamma}_{\eta\sigma} + \Gamma^{\nu}_{\eta\sigma}\Gamma^{\gamma}_{\rho\gamma} + \Gamma^{\nu}_{\sigma\gamma}\Gamma^{\gamma}_{\rho\eta} + \Gamma^{\nu}_{\rho\eta}\Gamma^{\gamma}_{\sigma\gamma}\right) + \\ & + g^{\nu\rho}g^{\eta\sigma}\left(\Gamma^{\mu}_{\rho\gamma}\Gamma^{\gamma}_{\eta\sigma} + \Gamma^{\mu}_{\eta\sigma}\Gamma^{\gamma}_{\rho\gamma} + \Gamma^{\mu}_{\sigma\gamma}\Gamma^{\gamma}_{\rho\eta} + \Gamma^{\mu}_{\rho\eta}\Gamma^{\gamma}_{\sigma\gamma}\right) + \\ & + g^{\rho\eta}g^{\sigma\gamma}\left(\Gamma^{\mu}_{\rho\sigma}\Gamma^{\nu}_{\eta\gamma} - \Gamma^{\mu}_{\rho\eta}\Gamma^{\nu}_{\sigma\gamma}\right)\Big]. \end{aligned} \tag{74}$$

It is possible to find in a curved space-time a coordinate system such that $t^{\mu\nu} = 0$. Similarly, an election of curvilinear coordinates in a flat space-time can yield non-vanishing values for the components of $t^{\mu\nu}$. We infer from this that the energy of the gravitational field is a global property, not a local one. There is energy in a region where there is a gravitational field, but in General Relativity it makes no sense to talk about the energy of a given point of the field.

8. Weyl tensor and the entropy of gravitation

The Weyl curvature tensor is the traceless component of the curvature (Riemann) tensor. In other words, it is a tensor that has the same symmetries as the Riemann curvature tensor with the extra condition that metric contraction yields zero.

In dimensions 2 and 3 the Weyl curvature tensor vanishes identically. In dimensions ≥ 4, the Weyl curvature is generally nonzero.

If the Weyl tensor vanishes, then there exists a coordinate system in which the metric tensor is proportional to a constant tensor.

The Weyl tensor can be obtained from the full curvature tensor by subtracting out various traces. This is most easily done by writing the Riemann tensor as a (0, 4)-valent tensor (by contracting with the metric). The Riemann tensor has 20 independent components, 10 of which are given by the Ricci tensor and the remaining 10 by the Weyl tensor.

The Weyl tensor is given in components by

$$C_{abcd} = R_{abcd} + \frac{2}{n-2}(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) + \frac{2}{(n-1)(n-2)} R\, g_{a[c}g_{d]b}, \tag{75}$$

where R_{abcd} is the Riemann tensor, R_{ab} is the Ricci tensor, R is the Ricci scalar and [] refers to the antisymmetric part. In 4 dimensions the Weyl tensor is:

$$C_{abcd} = R_{abcd} + \frac{1}{2}\left(g_{ad}R_{cb} + g_{bc}R_{da} - g_{ac}R_{db} - g_{bd}R_{ca}\right) + \frac{1}{6}\left(g_{ac}g_{db} - g_{ad}g_{cb}\right) R. \tag{76}$$

In addition to the symmetries of the Riemann tensor, the Weyl tensor satisfies

$$C^a_{bad} \equiv 0. \tag{77}$$

Two metrics that are *conformally related* to each other, i.e.

$$\bar{g}_{ab} = \Omega^2 g_{ab}, \tag{78}$$

where $\Omega(x)$ is a non-zero differentiable function, have the same Weyl tensor:

$$\bar{C}^a_{bcd} = C^a_{bcd}. \tag{79}$$

The absence of structure in space-time (i.e. spatial isotropy and hence no gravitational principal null-directions) corresponds to the absence of Weyl conformal curvature ($C^2 = C^{abcd}C_{abcd} = 0$). When clumping takes place, the structure is characterized by a non-zero Weyl curvature. In the interior of a black hole the Weyl curvature is large and goes to infinity at the singularity. Actually, Weyl curvature goes faster to infinity than Riemann curvature (the first as r^{-3} and the second as $r^{-3/2}$ for a Schwarzschild black hole). Since the initial conditions of the Universe seem highly uniform and the primordial state one of low-entropy, Penrose (1979) has proposed that the Weyl tensor gives a measure of the gravitational entropy and that the Weyl curvature vanishes at any initial singularity (this would be valid for white holes if they were to exist). In this way, despite the fact that matter was in local equilibrium in the early Universe, the global state was of low entropy, since the gravitational field was highly uniform and dominated the overall entropy.

9. Gravitational waves

Before the development of General Relativity Lorentz had speculated that "gravitation can be attributed to actions which do not propagate with a velocity larger than that of the light" (Lorentz 1900). The term *gravitational waves* appeared by first time in 1905 when H. Poincaré discussed the extension of Lorentz invariance to gravitation (Poincaré 1905, see Pais 1982 for further details). The idea that a perturbation in the source of the gravitational field can result in a wave that would manifest as a moving disturbance in the metric field was developed by Einstein in 1916, shortly after the final formulation of the field equations (Einstein 1916). Then, in 1918, Einstein presented the quadrupole formula for the energy loss of a mechanical system (Einstein 1918).

Einstein's approach was based on the weak-field approximation of the metric field:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{80}$$

where $\eta_{\mu\nu}$ is the Minkowski flat metric and $|h_{\mu\nu}| << 1$ is a small perturbation to the background metric. Since $h_{\mu\nu}$ is small, all products that involves it and its derivatives can be neglected. And because of the metric is almost flat all indices can be lowered or raised through $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ instead of $g_{\mu\nu}$ and $g^{\mu\nu}$. We can then write:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{81}$$

With this, we can compute the affine connection:

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2}\eta^{\mu\beta}(h_{\sigma\beta,\nu} + h_{\nu\beta,\sigma} - h_{\nu\sigma,\beta}) = \frac{1}{2}(h^{\mu}_{\sigma,\nu} + h^{\mu}_{\nu,\sigma} - h_{\nu\sigma}^{;\mu}). \tag{82}$$

Here, we denote $\eta^{\mu\beta}h_{\nu\sigma,\beta}$ by $h_{\nu\sigma}^{;\mu}$. Introducing $h = h^{\mu}_{\mu} = \eta^{\mu\nu}h_{\mu\nu}$, we can write the Ricci tensor and curvature scalar as:

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} = \frac{1}{2}(h_{,\mu\nu} - h^{\alpha}_{\nu,\mu\alpha} - h^{\alpha}_{\mu,\nu\alpha} + h^{\alpha}_{\mu\nu,\alpha}), \tag{83}$$

and

$$R \equiv g^{\mu\nu}R_{\mu\nu} = \eta^{\mu\nu}R_{\mu\nu} = h^{\alpha}_{,\alpha} - h^{\alpha\beta}_{,\alpha\beta}. \tag{84}$$

Then, the field equations (35) can be cast in the following way:

$$\bar{h}^{\alpha}_{\mu\nu,\alpha} + (\eta_{\mu\nu}\bar{h}^{\alpha\beta}_{,\alpha\beta} - \bar{h}^{\alpha}_{\nu,\mu\alpha} - \bar{h}^{\alpha}_{\mu,\nu\alpha}) = 2kT_{\mu\nu}, \tag{85}$$

where

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}. \tag{86}$$

We can make further simplifications through a *gauge transformation*. A gauge transformation is a small change of coordinates

$$x'^{\mu} \equiv x^{\mu} + \xi^{\mu}(x^{\alpha}), \tag{87}$$

where the ξ^{α} are of the same order of smallness as the perturbations of the metric. The matrix $\Lambda^{\mu}_{\nu} \equiv \partial x'^{\mu}/\partial x^{\nu}$ is given by:

$$\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \xi^{\mu}_{,\nu}. \tag{88}$$

Under gauge transformations:

$$\bar{h}'^{\mu\nu} = \bar{h}^{\mu\nu} - \xi^{\mu,\nu} - \xi^{\nu,\mu} + \eta^{\mu\nu}\xi^{\alpha}_{,\alpha}. \tag{89}$$

The gauge transformation can be chosen in such a way that:

$$\bar{h}^{\mu\alpha}_{,\alpha} = 0. \tag{90}$$

Then, the field equations result simplified to:

$$\bar{h}^{\alpha}_{\mu\nu,\alpha} = 2\kappa T_{\mu\nu}. \tag{91}$$

The imposition of this gauge condition is analogous to what is done in electromagnetism with the introduction of the Lorentz gauge condition $A^{\mu}_{,\mu} = 0$ where A^{μ} is the electromagnetic 4-potential. A gauge transformation $A_{\mu} \rightarrow A_{\mu} - \psi_{,\mu}$ preserves the Lorentz gauge condition iff $\psi^{\mu}_{,\mu} = 0$. In the gravitational case, we have $\xi^{\mu\alpha}_{,\alpha} = 0$.

Introducing the d'Alembertian:

$$\Box^2 = \eta^{\mu\nu}\partial_{\mu}\partial_{\nu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2, \tag{92}$$

we get:

$$\Box^2\bar{h}^{\mu\nu} = 2\kappa T^{\mu\nu}, \tag{93}$$

if $\bar{h}^{\mu\nu}_{,\nu} = 0$.

The gauge condition can be expressed as:

$$\Box^2\xi^{\mu} = 0. \tag{94}$$

Reminding the definition of κ we can write the wave equations of the gravitational field, insofar the amplitudes are small, as:

$$\Box^2\bar{h}^{\mu\nu} = -\frac{16\pi G}{c^4}T^{\mu\nu}. \tag{95}$$

In the absence of matter and non-gravitational fields, these equations become:

$$\Box^2\bar{h}^{\mu\nu} = 0. \tag{96}$$

The simplest solution to Eq. (96) is:

$$\bar{h}^{\mu\nu} = \Re[A^{\mu\nu}\exp{(ik_{\alpha}x^{\alpha})}], \tag{97}$$

where $A^{\mu\nu}$ is the amplitude matrix of a plane wave that propagates with direction $k^\mu = \eta^{\mu\alpha}k_\alpha$ and $\Re$ indicates that just the real part of the expression should be considered. The 4-vector k^μ is null and satisfies:

$$A^{\mu\nu}k_\nu = 0. \tag{98}$$

Since $h^{\bar{\mu}\nu}$ is symmetric the amplitude matrix has ten independent components. Equation (98) can be used to reduce this number to six. The gauge condition allows a further reduction, so finally we have only two independent components. Einstein realized of this in 1918. These two components characterize two different possible polarization states for the gravitational waves. In the so-called *traceless and transverse gauge* –TT– we can introduce two linear polarization matrices defined as:

$$e_1^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{99}$$

and

$$e_2^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{100}$$

in such a way that the general amplitude matrix is:

$$A^{\mu\nu} = \alpha e_1^{\mu\nu} + \beta e_2^{\mu\nu}, \tag{101}$$

with α and β complex constants.

The general solution of Eq. (95) is:

$$\bar{h}^{\mu\nu}(x^0,\ \vec{x}) = \frac{\kappa}{2\pi}\int \frac{T^{\mu\nu}(x^0 - |\vec{x} - \vec{x}'|\,,\ \vec{x}')}{|\vec{x} - \vec{x}'|}dV'. \tag{102}$$

In this integral we have considered only the effects of sources in the *past* of the space-time point $(x^0,\ \vec{x})$. The integral extends over the space-time region formed by the intersection the past half of the null cone at the field point with the world tube of the source. If the source is small compared to the wavelength of the gravitational radiation we can approximate (102) by:

$$\bar{h}^{\mu\nu}(ct,\ \vec{x}) = \frac{4G}{c^4 r}\int T^{\mu\nu}(ct - r,\ \vec{x}')dV'. \tag{103}$$

This approximation is valid in the *far zone*, where $r > l$, with l the typical size of the source. In the case of a slowly moving source $T^{00} \approx \rho c^2$, with ρ the proper energy density. Then, the expression for $\bar{h}^{\mu\nu}$ can be written as:

$$\bar{h}^{ij}(ct,\ \vec{x}) \approx \frac{2G}{c^4 r}\frac{d^2}{dt^2}\int \rho x^i x^j dV|_{\rm ret}. \tag{104}$$

The notation indicates that the integral is evaluated at the retarded time $t - r/c$.

The gravitational power of source with moment of inertia I and angular velocity ω is:

$$\frac{dE}{dt} = \frac{32GI^2\omega^6}{5c^5}. \tag{105}$$

This formula is obtained considering the energy-momentum carried by the gravitational wave, which is quadratic in $h^{\mu\nu}$. In order to derive it, the weak field approximation must be abandoned. See Landau and Lifshitz (1962).

10. Alternative theories of gravitation

10.1. Scalar-tensor gravity

Perhaps the most important alternative theory of gravitation is the Brans-Dicke theory of scalar-tensor gravity (Brans & Dicke 1961). The original motivation for this theory was to implement the idea of Mach that the phenomenon of inertia was due to the acceleration of a given system respect to the general mass distribution of the universe. The masses of the different fundamental particles would not be basic intrinsic properties but a relational property originated in the interaction with some cosmic field. We can express this in the form:

$$m_i(x^\mu) = \lambda_i \phi(x^\mu).$$

Since the masses of the different particles can be measured only through the gravitational acceleration Gm/r^2, the gravitational constant G should be related to the average value of some cosmic scalar field ϕ, which is coupled with the mass density of the universe.

The simplest general covariant equation for a scalar field produced by matter is:

$$\Box^2\phi = 4\pi\lambda (T^{\mathrm{M}})^{\mu}_{\mu}, \tag{106}$$

where $\Box^2 = \phi;^{\mu};_{\mu}$ is, again, the invariant d'Alembertian, λ is a coupling constant, and $(T^{\mathrm{M}})^{\mu\nu}$ is the energy-momentum of everything but gravitation. The matter and non-gravitational fields generate the cosmic scalar field ϕ. This field is normalized such that:

$$\langle\phi\rangle = \frac{1}{G}. \tag{107}$$

The scalar field, as anything else, also generates gravitation, so the Einstein field equations are re-written as:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi}{c^4\phi}\left(T^{\mathrm{M}}_{\mu\nu} + T^{\phi}_{\mu\nu}\right). \tag{108}$$

Here, $T^{\phi}_{\mu\nu}$ is the energy momentum tensor of the scalar field ϕ. Its explicit form is rather complicated (see Weinberg 1972, p. 159). Because of historical reasons the parameter λ is written as:

$$\lambda = \frac{2}{3+2\omega}.$$

In the limit $\omega \to \infty$, $\lambda \to 0$ and $T^{\phi}_{\mu\nu}$ vanishes, and hence the Brans-Dicke theory reduces to Einstein's.

One of the most interesting features of Brans-Dicke theory is that G varies with time because it is determined by the scalar field ϕ. A variation of G would affect the orbits of planets, the stellar evolution, and many other astrophysical phenomena. Experiments can constrain ω to $\omega > 500$. Hence, Einstein theory seems to be correct, at least at low energies.

10.2. Gravity with large extra dimensions

The so-called hierarchy problem is the difficulty to explain why the characteristic energy scale of gravity, the Planck energy: $M_{\rm P}c^2 \sim 10^{19}$ GeV[4], is 16 orders of magnitude larger than the electro-weak scale, $M_{\rm e-w}c^2 \sim 1$ TeV. A possible solution was presented in 1998 by Arkani-Hamed et al., with the introduction of gravity with large extra dimensions (LEDs). The idea of extra-dimensions was, however, no new in physics. It was originally introduced by Kaluza (1921) with the aim of unifying gravitation and electromagnetism. In a different context, Nordstrøm (1914) also discussed the possibility of a fifth dimension.

Kaluza's fundamental insight was to write the action as:

$$S = \frac{1}{16\pi\hat{G}}\int_{\mathcal{R}} \hat{R}\sqrt{-\hat{g}}\; d^4x dy, \tag{109}$$

instead of in the form given by expression (65). In Kaluza's action y is the coordinate of an extra dimension and the hats denote 5-dimensional (5-D) quantities. The interval results:

$$ds^2 = \hat{g}_{\mu\nu}dx^{\mu}dx^{\nu}, \tag{110}$$

with μ, ν running from 0 to 4, being $x^4 = y$ the extra dimension. Since the extra dimension should have no effect over the gravitation Kaluza imposed the condition:

$$\frac{\partial \hat{g}_{\mu\nu}}{\partial y} = 0. \tag{111}$$

Since gravitation manifests through the derivatives of the metric the condition (111) implies that the extra dimension does not affect the predictions of general relativity. If we write the metric as:

$$\hat{g}_{\mu\nu} = \phi^{-1/3}\begin{pmatrix} g_{\mu\nu} + \phi A_{\mu}A_{\nu} & \phi A_{\mu} \\ \phi A_{\nu} & \phi \end{pmatrix}. \tag{112}$$

Then, the action becomes

$$S = \frac{1}{16\pi G}\int_{\mathcal{R}} \left(R - \frac{1}{4}\phi F_{ab}F^{ab} - \frac{1}{6\phi^2 \partial_a \phi}\partial^a \phi \right) \sqrt{-\hat{g}}\; d^4x, \tag{113}$$

---

[4] The Planck mass is $M_{\rm P} = \sqrt{\hbar c/G} = 2.17644(11) \times 10^{-5}$ g. The Planck mass is the mass of the Planck particle, a hypothetical minuscule black hole whose Schwarzschild radius equals the Planck length ($l_{\rm P} = \sqrt{\hbar G/c^3} = 1.616?252(81) \times 10^{-33}$ cm).

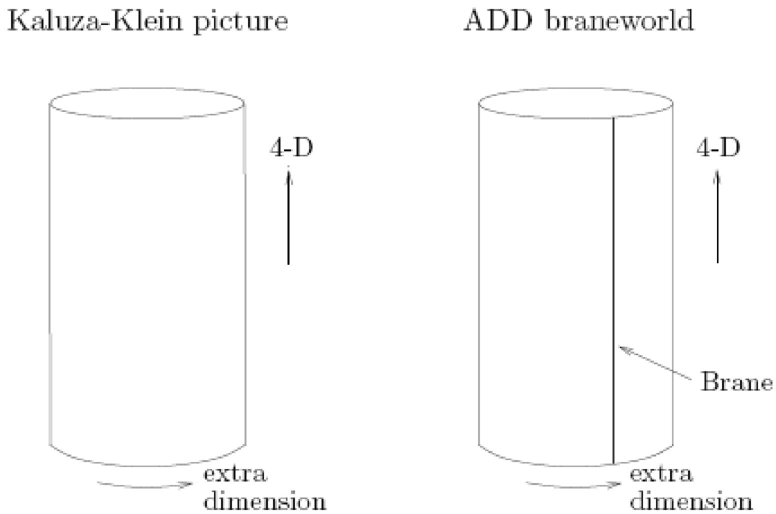

Figure 4. Compactified extra dimensions Kaluza-Klein and ADD braneworld theories. Adapted from Whisker (2006).

where $F_{ab} = \partial_a A_b - \partial_b A_a$ and

$$G = \frac{\hat{G}}{\int dy}.$$

The action (113) describes 4-D gravity along with electromagnetism. The price paid for this unification was the introduction of a scalar field ϕ called the *dilaton* (which was fixed by Kaluza $\phi = 1$) and an extra fifth dimension which is not observed.

Klein (1926) suggested that the fifth dimension was not observable because it is *compactified* on a circle. This compactification can be achieved identifying y with $y + 2\pi R$. The quantity R is the size of the extra dimension. Such a size should be extremely small in order not to be detected in experiments. The only natural length of the theory is the Planck length: $R \approx l_{\rm P} \sim 10^{-35}$ m.

Unfortunately, the Kaluza-Klein theory is not consistent with other observed features of particle physics as described by the Standard Model. This shortcoming is removed in the mentioned LED model by Arkani-Hamed et at. (1998), called ADD braneworld model. The model postulates n flat, compact extra dimensions of size R, but the Standard Model fields are *confined* to a 4-D *brane*, with only gravity propagating in the *bulk* (see Fig. 4). The effective potential for gravity behaves as[5]:

$$V(r) \approx \frac{m_1 m_2}{M_{\rm f}^{2+n}} \frac{1}{r^{n+1}}, \quad r \ll R, \tag{114}$$

[5]Notice that $G = M_{\rm P}^{-2}\hbar c$ or $G = M_{\rm P}^{-2}$ in units of $\hbar c = 1$.

$$V(r) \approx \frac{m_1 m_2}{M_{\rm f}^{2+n}} \frac{1}{R^n r}, \quad r \gg R, \tag{115}$$

where $M_{\rm f}$ is the fundamental mass scale of gravity in the full $(4+n)$-D space-time. Hence, in the brane the *effective* 4-D Planck scale is given by:

$$M_{\rm P}^2 = M_{\rm f}^{2+n} R^n. \tag{116}$$

In this way the fundamental scale $M_{\rm f}$ can be much lower than the Planck mass. If the fundamental scale is comparable to the electroweak scale, $M_{\rm f}c^2 \sim M_{\rm e-w}c^2 \sim$ 1 TeV, then we have that $n \geq 2$.

Randall and Sundrum (1999) suggested that the bulk geometry might be curved and the brane could have a tension. Hence, the brane becomes a gravitating object, interacting dynamically with the bulk. A Randall-Sundrum (RS) universe consists of two branes of torsion σ_1 and σ_2 bounding a slice of an anti-de Sitter space[6]. The two branes are separated by a distance L and the fifth dimension y is periodic with period $2L$. The bulk Einstein equations read:

$$R_{ab} - \frac{1}{2} R g_{ab} = \Lambda_5 g_{ab}, \tag{117}$$

where the bulk cosmological constant Λ_5 can be expressed in terms of the curvature length l as:

$$\Lambda_5 = \frac{6}{l^2}. \tag{118}$$

The metric is:

$$ds^2 = a^2(y) \eta_{\mu\nu} dx^\mu dx^\nu + dy^2. \tag{119}$$

Using the previous expressions, we can write the metric as:

$$ds^2 = e^{-2|y|/l} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2. \tag{120}$$

The term $e^{-2|y|/l}$ is called the *warp* factor. The effective Planck mass becomes:

$$M_{\rm P}^2 = e^{2L/l} M_{\rm f}^3 l. \tag{121}$$

According to the ratio L/l, the effective Planck mass can change. If we wish to have $M_{\rm f}c^2 \sim 1$ TeV, then we need $L/l \sim 50$ in order to generate the observed Planck mass $M_{\rm f}c^2 \sim 10^{19}$ GeV.

Other RS universes consist of a single, positive tension brane immersed in an infinite (non-compact) extra dimension. The corresponding metric remains the same:

$$ds^2 = e^{-2|y|/l} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2.$$

The 5-D graviton propagates through the bulk, but only the zero (massless) mode moves on the brane (for details see Maartens 2004).

---

[6] An anti-de Sitter space-time has a metric that is a maximally symmetric vacuum solution of Einstein's field equations with an attractive cosmological constant (corresponding to a negative vacuum energy density and positive pressure). This space-time has a constant negative scalar curvature.

10.3. $f(R)$-Gravity

In f(R) gravity, the Lagrangian of the Einstein-Hilbert action:

$$S[g] = \int \frac{1}{2\kappa} R\sqrt{-g}\,\mathrm{d}^4x \tag{122}$$

is generalized to

$$S[g] = \int \frac{1}{2\kappa} f(R)\sqrt{-g}\,\mathrm{d}^4x, \tag{123}$$

where g is the determinant of the metric tensor $g \equiv |g_{\mu\nu}|$ and $f(R)$ is some function of the scalar (Ricci) curvature.

The field equations are obtained by varying with respect to the metric. The variation of the determinant is:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}.$$

The Ricci scalar is defined as

$$R = g^{\mu\nu}R_{\mu\nu}.$$

Therefore, its variation with respect to the inverse metric $g^{\mu\nu}$ is given by

$$\begin{aligned} \delta R &= R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \\ &= R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}(\nabla_\rho \delta\Gamma^\rho_{\nu\mu} - \nabla_\nu \delta\Gamma^\rho_{\rho\mu}) \end{aligned} \tag{124}$$

Since $\delta\Gamma^\lambda_{\mu\nu}$ is actually the difference of two connections, it should transform as a tensor. Therefore, it can be written as

$$\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda a}\left(\nabla_\mu \delta g_{a\nu} + \nabla_\nu \delta g_{a\mu} - \nabla_a \delta g_{\mu\nu}\right),$$

and substituting in the equation above:

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}.$$

The variation in the action reads:

$$\begin{aligned} \delta S[g] &= \frac{1}{2\kappa}\int \left(\delta f(R)\sqrt{-g} + f(R)\delta\sqrt{-g}\right)\,\mathrm{d}^4x \\ &= \frac{1}{2\kappa}\int \left(F(R)\delta R\sqrt{-g} - \frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} f(R)\right)\,\mathrm{d}^4x \\ &= \frac{1}{2\kappa}\int \sqrt{-g}\left(F(R)(R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}) - \frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu} f(R)\right)\,\mathrm{d}^4x, \end{aligned}$$

where $F(R) = \frac{\partial f(R)}{\partial R}$. Doing integration by parts on the second and third terms we get:

$$\delta S[g] = \frac{1}{2\kappa}\int \sqrt{-g}\delta g^{\mu\nu}\left(F(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + [g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu]F(R)\right)\,\mathrm{d}^4x.$$

By demanding that the action remains invariant under variations of the metric, i.e. $\delta S[g] = 0$, we obtain the field equations:

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + \left[g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right]F(R) = \kappa T_{\mu\nu}, \tag{125}$$

where $T_{\mu\nu}$ is the energy-momentum tensor defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_{\rm m})}{\delta g^{\mu\nu}},$$

and $L_{\rm m}$ is the matter Lagrangian. If $F(R) = 1$, i.e. $f(R) = R$ we recover Einstein's theory.

11. What is a star?

The idea that stars are self-gravitating gaseous bodies was introduced in th XIX Century by Lane, Kelvin and Helmholtz. They suggested that stars should be understood in terms of the equation of hydrostatic equilibrium:

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}, \tag{126}$$

where the pressure P is given by

$$P = \frac{\rho kT}{\mu m_p}. \tag{127}$$

Here, k is Boltzmann's constant, μ is the mean molecular weight, T the temperature, ρ the mass density, and m_p the mass of the proton. Kelvin and Helmholtz suggested that the source of heat was due to the gravitational contraction. However, if the luminosity of a star like the Sun is taken into account, the total energy available would be released in 10^7 yr, which is in contradiction with the geological evidence that can be found on Earth. Eddington made two fundamental contributions to the theory of stellar structure proposing i) that the source of energy was thermonuclear reactions and ii) that the outward pressure of radiation should be included in Eq. (126). Then, the basic equations for stellar equilibrium become (Eddington 1926):

$$\frac{d}{dr}\left[\frac{\rho kT}{\mu m_p} + \frac{1}{3}aT^4\right] = -\frac{GM(r)\rho(r)}{r^2}, \tag{128}$$

$$\frac{dP_{\rm rad}(r)}{dr} = -\left(\frac{L(r)}{4\pi r^2 c}\right)\frac{1}{l}, \tag{129}$$

$$\frac{dL(r)}{dr} = 4\pi r^2\epsilon\rho, \tag{130}$$

where l is the mean free path of the photons, L the luminosity, and ϵ the energy generated per gram of material per unit time.

Once the nuclear power of the star is exhausted, the contribution from the radiation pressure decreases dramatically when the temperature diminishes. The star then contracts until a new pressure helps to balance gravity attraction: the degeneracy pressure of the electrons. The equation of state for a degenerate gas of electrons is:

$$P_{\rm rel} = K\rho^{4/3}. \tag{131}$$

Then, using Eq. (126),

$$\frac{M^{4/3}}{r^5} \propto \frac{GM^2}{r^5}. \tag{132}$$

Since the radius cancels out, this relations can be satisfied by a unique mass:

$$M = 0.197 \left[\left(\frac{hc}{G}\right)^3 \frac{1}{m_p^2}\right] \frac{1}{\mu_e^2} = 1.4 \;\; M_\odot, \tag{133}$$

where μ_e is the mean molecular weight of the electrons. The result implies that a completely degenerated star has this and only this mass. This limit was found by Chandrasekhar (1931) and is known as the *Chandrasekhar limit*.

In 1939 Chandrasekhar conjectured that massive stars could develop a degenerate core. If the degenerate core attains sufficiently high densities the protons and electrons will combine to form neutrons. “This would cause a sudden diminution of pressure resulting in the collapse of the star to the neutron core giving rise to an enormous liberation of gravitational energy. This may be the origin of the supernova phenomenon.” (Chandrasekhar 1939). An implication of this prediction is that the masses of neutron stars (objects supported by the degeneracy pressure of nucleons) should be close to 1.4 $M_\odot$, the maximum mass for white dwarfs. Not long before, Baade and Zwicky commented: “With all reserve we advance the view that supernovae represent the transitions from ordinary stars into *neutron stars* which in their final stages consist of extremely closely packed neutrons”. In this single paper Baade and Zwicky not only invented neutron stars and provided a theory for supernova explosions, but also proposed the origin of cosmic rays in these explosions (Baade & Zwicky 1934).

In the 1930s, neutron stars were not taken as a serious physical possibility. Oppenheimer & Volkoff (1939) concluded that if the neutron core was massive enough, then “either the Fermi equation of state must fail at very high densities, or the star will continue to contract indefinitely never reaching equilibrium”. In a subsequent paper Oppenheimer & Snyder (1939) chose between these two possibilities: “when all thermonuclear sources of energy are exhausted a sufficiently heavy star will collapse. This contraction will continue indefinitely till the radius of the star approaches asymptotically its gravitational radius. Light from the surface of the star will be progressively reddened and can escape over a progressively narrower range of angles till eventually the star tends to close itself off from any communication with a distant observer”. What we now understand for a black hole was then conceived. The scientific community paid no attention to these results, and Oppenheimer and many other scientists turned their efforts to win a war.

Black stars: an historical note

It is usual in textbooks to credit for the idea of black holes to John Michell and Pierre-Simon Laplace, in the XVIII Century. The idea of a body so massive that even light could not escape was put forward by geologist Rev. John Michell in a letter written to Henry Cavendish in 1783 to the Royal Society: "If the semi-diameter of a sphere of the same density as the Sun were to exceed that of the Sun in the proportion of 500 to 1, a body falling from an infinite height toward it would have acquired at its surface greater velocity than that of light, and consequently supposing light to be attracted by the same force in proportion to its inertia, with other bodies, all light emitted from such a body would be made to return toward it by its own proper gravity." (Michell 1783).

In 1796, the mathematician Pierre-Simon Laplace promoted the same idea in the first and second editions of his book *Exposition du système du Monde* (it was removed from later editions). Such "dark stars" were largely ignored in the nineteenth century, since light was then thought to be a massless wave and therefore not influenced by gravity. Unlike the modern black hole concept, the object behind the horizon is assumed to be stable against collapse. Moreover, no equation of state was adopted neither by Michell nor Laplace. Hence, their dark stars where Newtonian objects, infinitely rigid, and they have nothing to do with the nature of space and time, which were for the absolute concepts. Nonetheless, they could calculate correctly the size of such objects form the simple devise of equating the potential and escape energy from a body of mass M:

$$\frac{1}{2}mv^2 = \frac{GMm}{r^2}. \tag{134}$$

Just setting $v = c$ and assuming the gravitational and the inertial mass are the same, we get:

$$r_{\text{black star}} = \sqrt{\frac{2GM}{c^2}}. \tag{135}$$

12. Schwarzschild black holes

The first exact solution of Einstein field equations was found by Karl Schwarzschild in 1916. This solution describes the geometry of space-time outside a spherically symmetric matter distribution.

The most general spherically symmetric metric is:

$$ds^2 = \alpha(r,\ t)dt^2 - \beta(r,\ t)dr^2 - \gamma(r,\ t)d\Omega^2 - \delta(r,\ t)drdt, \tag{136}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. We are using spherical polar coordinates. The metric (136) is invariant under rotations (isotropic).

The invariance group of general relativity is formed by the group of general transformations of coordinates of the form $x'^{\mu} = f^{\mu}(x)$. This yields 4 degrees of freedom, two of which have been used when adopting spherical coordinates (the transformations that do not break the central symmetry are: $r' = f_1(r,\ t)$ and $t' = f_2(r,\ t)$). With the two available degrees of freedom we can freely choose two metric coefficients, whereas the other two are determined by Einstein equations. Some possibilities are:

- *Standard gauge.*

$$ds^2 = c^2 A(r,\ t)dt^2 - B(r,\ t)dr^2 - r^2 d\Omega^2.$$

- *Synchronous gauge.*

$$ds^2 = c^2 dt^2 - F^2(r,\ t)dr^2 - R^2(r,\ t)d\Omega^2.$$

- *Isotropic gauge.*

$$ds^2 = c^2 H^2(r,\ t)dt^2 - K^2(r,\ t)\left[dr^2 + r^2(r,\ t)d\Omega^2\right].$$

- *Co-moving gauge.*

$$ds^2 = c^2 W^2(r,\ t)dt^2 - U(r,\ t)dr^2 - V(r,\ t)d\Omega^2.$$

Adopting the standard gauge and a static configuration (no dependence of the metric coefficients on t), we can get equations for the coefficients A and B of the standard metric:

$$ds^2 = c^2 A(r)dt^2 - B(r)dr^2 - r^2 d\Omega^2. \tag{137}$$

Since we are interested in the solution *outside* the spherical mass distribution, we only need to require the Ricci tensor to vanish:

$$R_{\mu\nu} = 0.$$

According to the definition of the curvature tensor and the Ricci tensor, we have:

$$R_{\mu\nu} = \partial_\nu \Gamma^\sigma_{\mu\sigma} - \partial_\sigma \Gamma^\sigma_{\mu\nu} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} = 0. \tag{138}$$

If we remember that the affine connection depends on the metric as

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}),$$

we see that we have to solve a set of differential equations for the components of the metric $g_{\mu\nu}$.

The metric coefficients are:

$$\begin{aligned}
g_{00} &= c^2 A(r),\\
g_{11} &= -B(r),\\
g_{22} &= -r^2,\\
g_{33} &= -r^2 \sin^2\theta,\\
g^{00} &= 1/A(r),\\
g^{11} &= -1/B(r),\\
g^{22} &= -1/r^2,\\
g^{33} &= -1/r^2 \sin^2\theta.
\end{aligned}$$

Then, only nine of the 40 independent connection coefficients are different from zero. They are:

$$
\begin{aligned}
\Gamma^1_{01} &= A'/(2A), \\
\Gamma^1_{22} &= -r/B, \\
\Gamma^2_{33} &= -\sin\theta\cos\theta, \\
\Gamma^1_{00} &= A'/(2B), \\
\Gamma^1_{33} &= -(r\sin^2/B), \\
\Gamma^3_{13} &= 1/r, \\
\Gamma^1_{11} &= B'/(2B), \\
\Gamma^2_{12} &= 1/r, \\
\Gamma^3_{23} &= \cot\theta, .
\end{aligned}
$$

Replacing in the expression for $R_{\mu\nu}$:

$$
\begin{aligned}
R_{00} &= -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}, \\
R_{11} &= \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}, \\
R_{22} &= \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right), \\
R_{33} &= R_{22}\sin^2\theta.
\end{aligned}
$$

The Einstein field equations for the region of empty space then become:

$$R_{00} = R_{11} = R_{22} = 0$$

(the fourth equation has no additional information). Multiplying the first equation by B/A and adding the result to the second equation, we get:

$$A'B + AB' = 0,$$

from which $AB =$ constant. We can write then $B = \alpha A^{-1}$. Going to the third equation and replacing B we obtain: $A + rA' = \alpha$, or:

$$\frac{d(rA)}{dr} = \alpha.$$

The solution of this equation is:

$$A(r) = \alpha\left(1 + \frac{k}{r}\right),$$

with k another integration constant. For B we get:

$$B = \left(1 + \frac{k}{r}\right)^{-1}.$$

If now we consider the Newtonian limit:

$$\frac{A(r)}{c^2} = 1 + \frac{2\Phi}{c^2},$$

with Φ the Newtonian gravitational potential: $\Phi = -GM/r$, we can conclude that

$$k = -\frac{2GM}{c^2}$$

and

$$\alpha = c^2.$$

Therefore, the Schwarzschild solution for a static mass M can be written in spherical coordinates (t, r, θ, ϕ) as:

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (139)$$

As mentioned, this solution corresponds to the vacuum region exterior to the spherical object of mass M. Inside the object, space-time will depend on the peculiarities of the physical object.

The metric given by Eq. (139) has some interesting properties. Let's assume that the mass M is concentrated at $r = 0$. There seems to be two singularities at which the metric diverges: one at $r = 0$ and the other at

$$r_{\rm Schw} = \frac{2GM}{c^2}. \quad (140)$$

$r_{\rm Schw}$ is know as the *Schwarzschild radius*. If the object of mass M is macroscopic, then $r_{\rm Schw}$ is inside it, and the solution does not apply. For instance, for the Sun $r_{\rm Schw} \sim 3$ km. However, for a point mass, the Schwarzschild radius is in the vacuum region and space-time has the structure given by (139). In general, we can write:

$$r_{\rm Schw} \sim 3 \left(\frac{M}{M_\odot}\right) \quad \text{km}.$$

It is easy to see that strange things occur close to $r_{\rm Schw}$. For instance, for the proper time we get:

$$d\tau = \left(1 - \frac{2GM}{rc^2}\right)^{1/2} dt, \quad (141)$$

or

$$dt = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2} d\tau, \quad (142)$$

When $r \longrightarrow \infty$ both times agree, so t is interpreted as the proper time measure from an infinite distance. As the system with proper time τ approaches to $r_{\rm Schw}$, dt tends to infinity according to Eq. (142). The object never reaches the Schwarszchild surface when seen by an infinitely distant observer. The closer the object is to the Schwarzschild radius, the slower it moves for the external observer.

A direct consequence of the difference introduced by gravity in the local time respect to the time at infinity is that the radiation that escapes from a given $r > r_{\rm Schw}$ will be redshifted when received by a distant and static observer. Since the frequency (and hence the energy) of the photon depend of the time interval, we can write, from Eq. (142):

$$\lambda_\infty = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2} \lambda. \tag{143}$$

Since the redshift is:

$$z = \frac{\lambda_\infty - \lambda}{\lambda}, \tag{144}$$

then

$$1 + z = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2}, \tag{145}$$

and we see that when $r \longrightarrow r_{\rm Schw}$ the redshift becomes infinite. This means that a photon needs infinite energy to escape from inside the region determined by $r_{\rm Schw}$. Events that occur at $r < r_{\rm Schw}$ are disconnected from the rest of the universe. Hence, we call the surface determined by $r = r_{\rm Schw}$ an *event horizon*. Whatever crosses the event horizon will never return. This is the origin of the expression "black hole", introduced by John A. Wheeler in the mid 1960s. The black hole is the region of space-time inside the event horizon. We can see what happens with the light cones as an event is closer to the horizon of a Schwarzschild black hole in Figure 5. The shape of the cones can be calculated from the metric (139) imposing the null condition $ds^2 = 0$. Then,

$$\frac{dr}{dt} = \pm\left(1 - \frac{2GM}{r}\right), \tag{146}$$

where we made $c = 1$. Notice that when $r \to \infty$, $dr/dt \to \pm 1$, as in Minkowski space-time. When $r \to 2GM$, $dr/dt \to 0$, and light moves along the surface $r = 2GM$. For $r < 2GM$, the sign of the derivative is inverted. The inward region of $r = 2GM$ is time-like for any physical system that has crossed the boundary surface.

What happens to an object when it crosses the event horizon?. According to Eq. (139), there is a singularity at $r = r_{\rm Schw}$. However, the metric coefficients can be made regular by a change of coordinates. For instance we can consider Eddington-Finkelstein coordinates. Let us define a new radial coordinate r_* such that radial null rays satisfy $d(ct \pm r_*) = 0$. Using Eq. (139) it can be shown that:

$$r_* = r + \frac{2GM}{c^2} \log\left|\frac{r - 2GM/c^2}{2GM/c^2}\right|.$$

Then, we introduce:

$$v = ct + r_*.$$

The new coordinate v can be used as a time coordinate replacing t in Eq. (139). This yields:

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)(c^2dt^2 - dr_*^2) - r^2 d\Omega^2$$

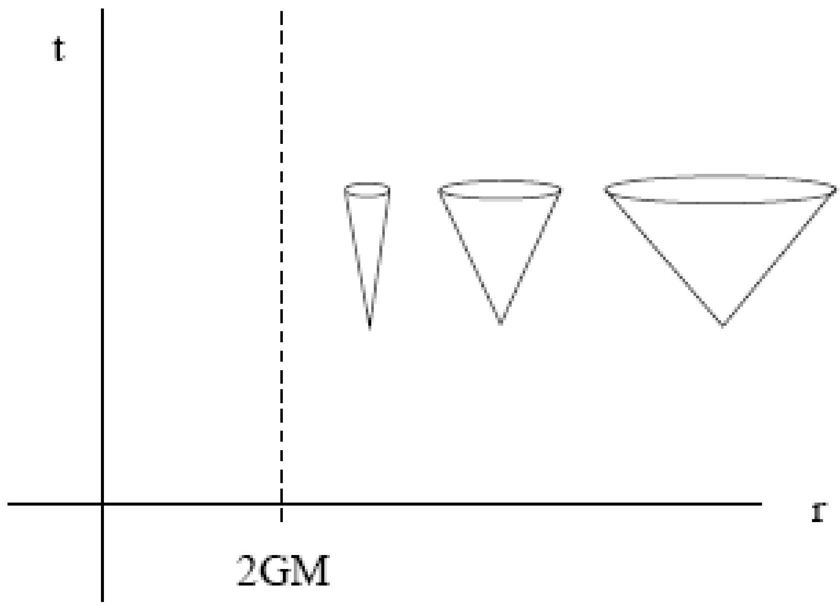

Figure 5. Space-time diagram in Schwarzschild coordinates showing the light cones of events at different distances of the event horizon (units $c = 1$). Adapted form Carroll (2003).

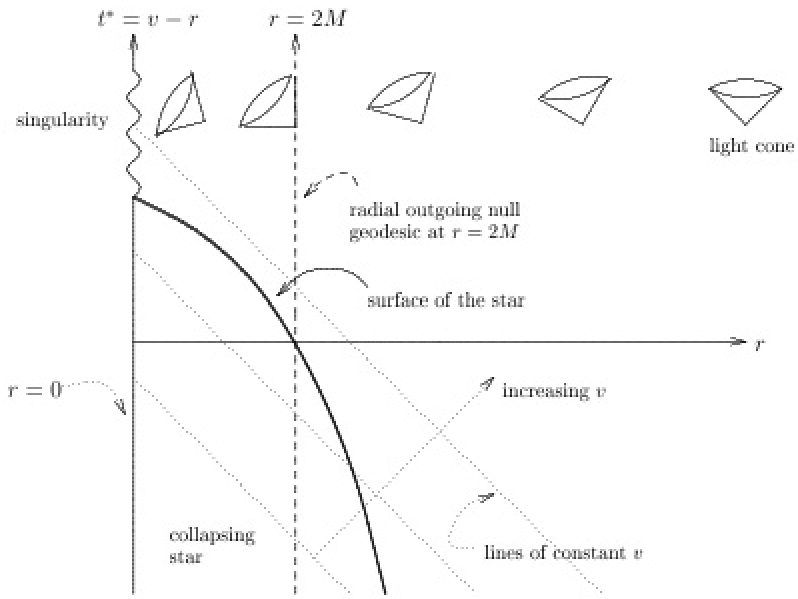

Figure 6. Space-time diagram in Eddington-Finkelstein coordinates showing the light cones close to and inside a black hole. Here, $r = 2M = r_{\rm Schw}$ is the Schwarzschild radius where the event horizon is located (units $G = c = 1$). Adapted form Townsend (1997).

or

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) dv^2 - 2drdv - r^2 d\Omega^2, \tag{147}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

Notice that in Eq. (147) the metric is non-singular at $r = 2GM/c^2$. The only real singularity is at $r = 0$, since there the Riemann tensor diverges. In order to plot the space-time in a $(t,\ r)$-plane, we can introduce a new time coordinate $t_* = v - r$. From the metric (147) or from Fig. 6 we see that the line $r = r_{\rm Schw}$, $\theta =$constant, and $\phi =$ constant is a null ray, and hence, the surface at $r = r_{\rm Schw}$ is a null surface. This null surface is an event horizon because inside $r = r_{\rm Schw}$ all cones have $r = 0$ in their future (see Figure 6). The object in $r = 0$ is the source of the gravitational field and is called *the singularity*. We will say more about it in Sect. 24. For the moment, we only remark that everything that crosses the event horizon will end at the singularity. This is the inescapable fate. There is no way to avoid it: in the future of every event inside the event horizon is the singularity. There is no escape, no hope, no freedom, inside the black hole. There is just the singularity, whatever such a thing might be.

We see now that the name "black hole" is not strictly correct for space-time regions isolated by event horizons. There is no hole to other place. What falls into the black hole, goes to the singularity. The singularity increases its mass and energy, and then the event horizon grows. This would not happen if what

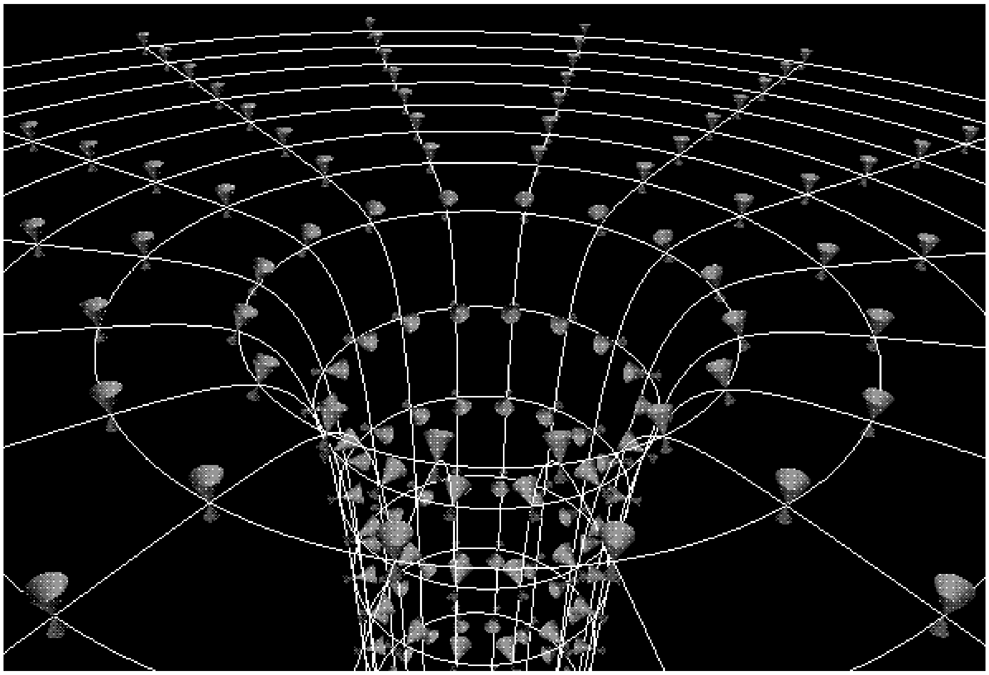

Figure 7. Embedding space-time diagram in Eddington-Finkelstein coordinates showing the light cones close of events at different distances from a Scharzschild black hole. From: www.faculty.iu-bremen.de/.../image030.gif.

falls into the hole were able to pass through, like through a hole in the wall. A black hole is more like a space-time precipice, deep, deadly, and with something unknown at the bottom. A graphic depiction with an embedding diagram of a Schwarzschild black hole is shown in Figure 7. An embedding is an immersion of a given manifold into a manifold of lower dimensionality that preserves the metric properties.

13. A General definition of black hole

We shall now provide a general definition of a black hole, independently of the coordinate system adopted in he description of space-time. First, we shall introduce some useful definitions (e.g. Hawking & Ellis 1973, Wald 1984).

Definition. *A causal curve in a space-time* $(M,\ g_{\mu\nu})$ *is a curve that is non space-like, that is, piecewise either time-like or null (light-like).*

We say that a given space-time $(M,\ g_{\mu\nu})$ is *time-orientable* if we can define over M a smooth non-vanishing time-like vector field.

Definition. *If* $(M,\ g_{\mu\nu})$ *is a time-orientable space-time, then* $\forall p \in M$, *the causal future of* p, *denoted* $J^{+}(p)$, *is defined by:*

$$J^{+}(p) \equiv \{q \in M | \exists \; a \; future - directed \; causal \; curve \; from \; p \; to \; q\}\,. \qquad (148)$$

Similarly,

Definition. *If $(M,\ g_{\mu\nu})$ is a time-orientable space-time, then $\forall p \in M$, the causal past of p, denoted $J^{-}(p)$, is defined by:*

$$J^{-}(p) \equiv \{q \in M | \exists \; a \; past - directed \; causal \; curve \; from \; p \; to \; q\}\,. \qquad (149)$$

Particle horizons occur whenever a particular system never gets to be influenced by the whole space-time. If a particle crosses the horizon, it will not exert any further action upon the system respect to which the horizon is defined.

Definition. *For a causal curve γ the associated future (past) particle horizon is defined as the boundary of the region from which the causal curves can reach some point on γ.*

Finding the particle horizons (if one exists at all) requires a knowledge of the global space-time geometry.

Let us now consider a space-time where all null geodesics that start in a region $\mathcal{J}^{-}$ end at $\mathcal{J}^{+}$. Then, such a space-time, $(M,\ g_{\mu\nu})$ is said to contain a *black hole* if M *is not* contained in $J^{-}(\mathcal{J}^{+})$. In other words, there is a region from where no null geodesic can reach the *asymptotic flat*[7] future space-time, or, equivalently, there is a region of M that is causally disconnected from the global future. The *black hole region*, BH, of such space-time is $BH = [M - J^{-}(\mathcal{J}^{+})]$, and the boundary of BH in M, $H = J^{-}(\mathcal{J}^{+}) \bigcap M$, is the *event horizon*.

14. Birkoff's theorem

If we consider the isotropic but *not static* line element,

$$ds^2 = c^2 A(r,\ t)dt^2 - B(r,\ t)dr^2 - r^2 d\Omega^2, \qquad (150)$$

and substitute into the Einstein empty-space field equations $R_{\mu\nu} = 0$ to obtain the functions $A(r,\ t)$ and $B(r,\ t)$, the result would be exactly the same:

$$A(r,\ t) = A(r) = \left(1 - \frac{2GM}{rc^2}\right),$$

and

$$B(r,\ t) = B(r) = \left(1 - \frac{2GM}{rc^2}\right)^{-1}.$$

[7] Asymptotic flatness is the property of a geometry of space-time which means that in appropriate coordinates, the limit of the metric at infinity approaches the metric of the flat (Minkowskian) space-time.

This result is general and known as the Birkoff's theorem:

The space-time geometry outside a general spherical symmetry matter distribution is the Schwarzschild geometry.

Birkoff's theorem implies that strictly radial motions do not perturb the space-time metric. In particular, a pulsating star, if the pulsations are strictly radial, does not produce gravitational waves.

The converse of Birkoff's theorem is not true, i.e.,

If the region of space-time is described by the metric given by expression (139), then the matter distribution that is the source of the metric does not need to be spherically symmetric.

15. Orbits

Orbits around a Schwarzschild black hole can be easily calculated using the metric and the relevant symmetries (see. e.g. Raine and Thomas 2005). Let us call k^μ a vector in the direction of a given symmetry (i.e. k^μ is a Killing vector). A static situation is symmetric in the time direction, hence we can write: $k^\mu = (1, 0, 0, 0)$. The 4-velocity of a particle with trajectory $x^\mu = x^\mu(\tau)$ is $u^\mu = dx^\mu/d\tau$. Then, since $u^0 = E/c$, where E is the energy, we have:

$$g_{\mu\nu} k^\mu u^\nu = g_{00} k^0 u^0 = g_{00} u^0 = \eta_{00} \frac{E}{c} = \frac{E}{c} = \text{ constant.} \tag{151}$$

If the particle moves along a geodesic in a Schwarzschild space-time, we obtain from Eq. (151):

$$c \left(1 - \frac{2GM}{c^2 r}\right) \frac{dt}{d\tau} = \frac{E}{c}. \tag{152}$$

Similarly, for the symmetry in the azimuthal angle ϕ we have $k^\mu = (0, 0, 0, 1)$, in such a way that:

$$g_{\mu\nu} k^\mu u^\nu = g_{33} k^3 u^3 = g_{33} u^3 = -L = \text{ constant.} \tag{153}$$

In the Schwarzschild metric we find, then,

$$r^2 \frac{d\phi}{d\tau} = L = \text{ constant.} \tag{154}$$

If now we divide the Schwarzschild interval (139) by $c^2 d\tau^2$:

$$1 = \left(1 - \frac{2GM}{c^2 r}\right) \left(\frac{dt}{d\tau}\right)^2 - c^{-2} \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 - c^{-2} r^2 \left(\frac{d\phi}{d\tau}\right)^2, \tag{155}$$

and using the conservation equations (152) and (154) we obtain:

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{c^2} - \left(c^2 + \frac{L^2}{r^2}\right) \left(1 - \frac{2GM}{c^2 r}\right). \tag{156}$$

If we express energy in units of m_0c^2 and introduce an effective potential $V_{\rm eff}$,

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{c^2} - V_{\rm eff}^2. \tag{157}$$

For circular orbits of a massive particle we have the conditions

$$\frac{dr}{d\tau} = 0, \quad \text{and} \quad \frac{d^2r}{d\tau^2} = 0.$$

The orbits are possible only at the turning points of the effective potential:

$$V_{\rm eff} = \sqrt{\left(c^2 + \frac{L^2}{r^2}\right)\left(1 - \frac{2r_{\rm g}}{r}\right)}, \tag{158}$$

where L is the angular momentum in units of m_0c and $r_{\rm g} = GM/c^2$ is the gravitational radius. Then,

$$r = \frac{L^2}{2cr_{\rm g}} \pm \frac{1}{2}\sqrt{\frac{L^4}{c^2r_{\rm g}^2} - 12L^2}. \tag{159}$$

The effective potential is shown in Figure 8 for different values of the angular momentum.

For $L^2 > 12c^2r_{\rm g}^2$ there are two solutions. The negative sign corresponds to a maximum of the potential and is unstable, and the positive sign corresponds to a minimum, which is stable. At $L^2 = 12c^2r_{\rm g}^2$ there is a single stable orbit. It is the innermost marginally stable orbit, and it occurs at $r = 6r_{\rm g} = 3r_{\rm Schw}$. The specific angular momentum of a particle in a circular orbit at r is:

$$L = c\left(\frac{r_{\rm g}r}{1 - 3r_{\rm g}/r}\right)^{1/2}.$$

Its energy (units of m_0c^2) is:

$$E = \left(1 - \frac{2r_{\rm g}}{r}\right)\left(1 - \frac{3r_{\rm g}}{r}\right)^{-1/2}.$$

The proper and observer's periods are:

$$\tau = \frac{2\pi}{c}\left(\frac{r^3}{r_{\rm g}}\right)^{1/2}\left(1 - \frac{3r_{\rm g}}{r}\right)^{1/2}$$

and

$$T = \frac{2\pi}{c}\left(\frac{r^3}{r_{\rm g}}\right)^{1/2}.$$

Notice that when $r \longrightarrow 3r_{\rm g}$ both L and E tend to infinity, so only massless particles can orbit at such a radius.

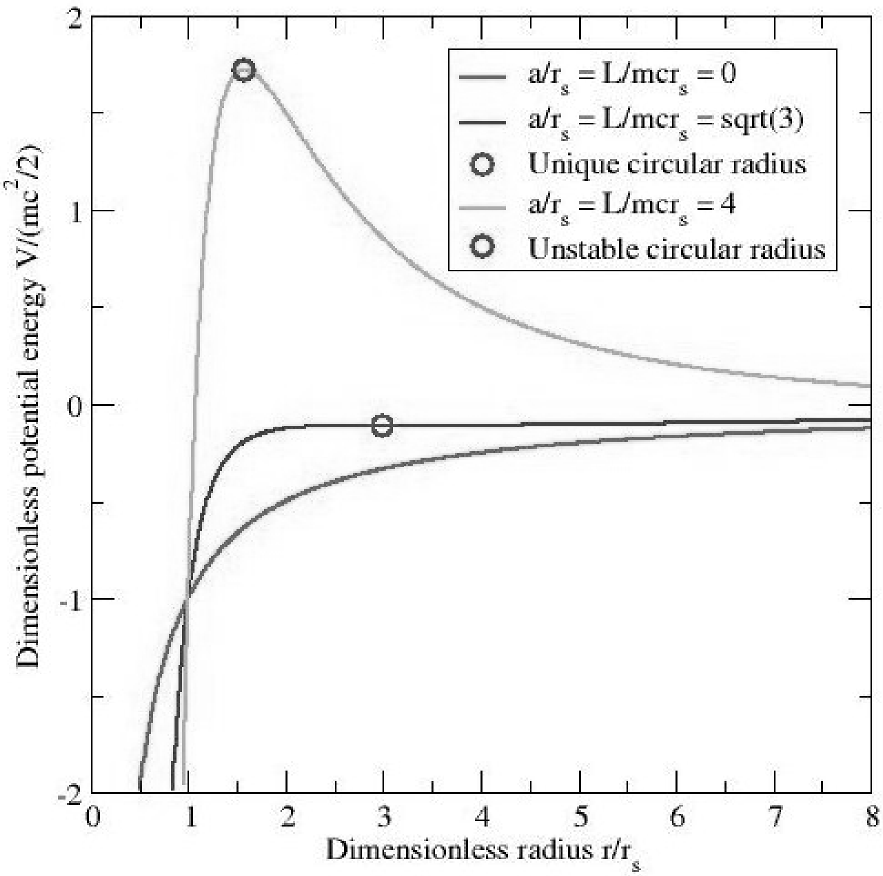

Figure 8. General relativistic effective potential plotted for several values of angular momentum. Copyright (C) 2000,2001,2002 Free Software Foundation, Inc.

The local velocity at r of an object falling from rest to the black hole is (e.g. Raine and Thomas 2005):

$$v_{\rm loc} = \frac{\text{proper distance}}{\text{proper time}} = \frac{dr}{(1 - 2GM/c^2 r)\, dt},$$

hence, using the expression for dr/dt from the metric (139):

$$\frac{dr}{dt} = -c\left(\frac{2GM}{c^2 r}\right)^{1/2}\left(1 - \frac{2GM}{c^2 r}\right), \tag{160}$$

we have,

$$v_{\rm loc} = \left(\frac{2r_{\rm g}}{r}\right)^{1/2} \quad \text{(in units of } c\text{)}. \tag{161}$$

Then, the differential acceleration the object will experience along an element dr is[8]:

$$dg = \frac{2r_{\rm g}}{r^3}c^2\, dr. \tag{162}$$

The tidal acceleration on a body of finite size Δr is simply $(2r_{\rm g}/r^3)c^2\,\Delta r$. This acceleration and the corresponding force becomes infinite at the singularity. As the object falls into the black hole, tidal forces act to tear it apart. This painful process is known as "spaghettification". The process can be significant long before crossing the event horizon, depending on the mass of the black hole.

The energy of a particle in the innermost stable orbit can be obtained from the above equation for the energy setting $r = 6r_{\rm g}$. This yields (unites of m_0c^2):

$$E = \left(1 - \frac{2r_{\rm g}}{6r_{\rm g}}\right)\left(1 - \frac{3r_{\rm g}}{6r_{\rm g}}\right)^{-1/2} = \frac{2}{3}\sqrt{2}.$$

Since a particle at rest at infinity has $E = 1$, then the energy that the particle should release to fall into the black hole is $1 - (2/3)\sqrt{2} = 0.057$. This means 5.7 % of its rest mass energy, significantly higher than the energy release that can be achieved through nuclear fusion.

An interesting question we can ask is what is the gravitational acceleration at the event horizon as seen by an observer from infinity. The acceleration relative to a hovering frame system of a freely falling object at rest at r is (Raine and Thomas 2005):

$$g_r = -c^2\left(\frac{GM/c^2}{r^2}\right)\left(1 - \frac{2GM/c^2}{r}\right)^{-1/2}.$$

So, the energy spent to move the object a distance dl will be $dE_r = mg_r dl$. The energy expended respect to a frame at infinity is $dE_\infty = mg_\infty dl$. Because of the conservation of energy, both quantities should be related by a redshift factor:

$$\frac{E_r}{E_\infty} = \frac{g_r}{g_\infty} = \left(1 - \frac{2GM/c^2}{r}\right)^{-1/2}.$$

[8]Notice that $dv_{\rm loc}/d\tau = (dv_{\rm loc}/dr)(dr/d\tau) = (dv_{\rm loc}/dr)v_{\rm loc} = r_{\rm g}c^2/r^2$.

Hence, using the expression for g_r we get:

$$g_\infty = c^2 \frac{GM/c^2}{r^2}. \tag{163}$$

Notice that for an observer at r, $g_r \longrightarrow \infty$ when $r \longrightarrow r_{\rm Schw}$. However, from infinity the required force to hold the object hovering at the horizon is:

$$mg_\infty = c^2 \frac{GmM/c^2}{r_{\rm Schw}^2} = \frac{mc^4}{4GM}.$$

This is the *surface gravity* of the black hole.

Radial motion of photons

For photons we have that $ds^2 = 0$. The radial motion, then, satisfies:

$$\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 = 0. \tag{164}$$

From here,

$$\frac{dr}{dt} = \pm c \left(1 - \frac{2GM}{rc^2}\right). \tag{165}$$

Integrating, we have:

$$\begin{aligned} ct &= r + \frac{2GM}{c^2} \ln \left| \frac{rc^2}{2GM} - 1 \right| + \text{ constant} \quad \text{outgoing photons,} && (166) \\ ct &= -r - \frac{2GM}{c^2} \ln \left| \frac{rc^2}{2GM} - 1 \right| + \text{ constant} \quad \text{incoming photons.} && (167) \end{aligned}$$

Notice that in a $(ct,\ r)$-diagram the photons have worldlines with slopes ± 1 as $r \to \infty$, indicating that space-time is asymptotically flat. As the events that generate the photons approach to $r = r_{\rm Schw}$, the slopes tend to $\pm\infty$. This means that the light cones become more and more thin for events close to the event horizon. At $r = r_{\rm Schw}$ the photons cannot escape and they move along the horizon (see Fig. 5). An observer in the infinity will never detect them.

Circular motion of photons

In this case, fixing $\theta =$constant due to the symmetry, we have that photons will move in a circle of $r =$constant and $ds^2 = 0$. Then, from (139), we have:

$$\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - r^2 d\phi^2 = 0. \tag{168}$$

This means that

$$\dot{\phi} = \frac{c}{r} \sqrt{\left(1 - \frac{2GM}{rc^2}\right)} = \text{ constant.}$$

The circular velocity is:

$$v_{\rm circ} = \frac{r\dot{\phi}}{\sqrt{g_{00}}} = \frac{\Omega r}{(1 - 2GM/c^2r)^{1/2}}. \tag{169}$$

Setting $v_{\rm circ} = c$ for photons and using $\Omega = (GM/r^3)^{1/2}$, we get that the only possible radius for a circular photon orbit is:

$$r_{\rm ph} = \frac{3GM}{c^2}. \tag{170}$$

For a compact object of 1 $M_{\odot}$, $r \approx 4.5$ km, in comparison with the Schwarzschild radius of 3 km. Photons moving at this distance form the "photosphere" of the black hole. The orbit, however, is unstable, as it can be seen from the effective potential:

$$V_{\rm eff} = \frac{L_{\rm ph}^2}{r^2}\left(1 - \frac{2r_{\rm g}}{r}\right). \tag{171}$$

Notice that the four-acceleration for circular motion is: $a_{\mu} = u_{\mu}u_{\nu};^{\nu}$. The radial component in the Schawrzschild metric is:

$$a_r = \frac{GM/r^2 - \Omega^2 r}{1 - 2GM/c^2r - \Omega^2 r^2/c^2}. \tag{172}$$

The circular motion along a geodesic line corresponds to the case $a_r = 0$ (free motion). This gives from Eq. (172) the usual expression for the Keplerian angular velocity

$$\Omega_{\rm K} = \left(\frac{GM}{r^3}\right)^{1/2},$$

already used in deriving $r_{\rm ph}$. In general, however, the angular velocity can have any value determined by the metric and can be quite different from the corresponding Keplerian value. In general:

$$v = \frac{r\Omega_{\rm K}}{(1 - 2GM/c^2r)^{1/2}} = \left(\frac{GM}{r}\right)^{1/2}\left(1 - \frac{2GM}{c^2r}\right)^{-1/2}. \tag{173}$$

From this latter equation and the fact that $v \leq c$ it can be concluded that pure Keplerian motion is only possible for $r \geq 1.5r_{\rm Schw}$. At $r \leq 1.5r_{\rm Schw}$ any massive particle will find its mass increased by special relativistic effects in such a way that the gravitational attraction will outweigh any centrifugal force.

Gravitational capture

A particle coming from infinity is captured if its trajectory ends in the black hole. The angular momentum of a non-relativistic particle with velocity v_{∞} at infinity is $L = mv_{\infty}b$, where b is an impact parameter. The condition $L/mcr_{\rm Schw} = 2$ defines $b_{\rm cr,\,non-rel} = 2r_{\rm Schw}(c/v_{\infty})$. Then, the capture cross section is:

$$\sigma_{\rm non-rel} = \pi b_{\rm cr}^2 = 4\pi\frac{c^2 r_{\rm Schw}^2}{v_{\infty}^2}. \tag{174}$$

For an ultra-relativistic particle, $b_{\rm cr} = 3\sqrt{3}r_{\rm Schw}/2$, and then

$$\sigma_{\rm rel} = \pi b_{\rm cr}^2 = \frac{27}{4}\pi r_{\rm Schw}^2. \tag{175}$$

16. Other coordinate systems

Other coordinates can be introduced to study additional properties of black holes. We refer the reader to the books of Frolov and Novikov (1998) and Raine and Thomas (2005) for further details. Here we shall only introduce the Kruskal-Szekeres coordinates. These coordinates have the advantage that they cover the entire space-time manifold of the maximally extended Schwarzschild solution and are well-behaved everywhere outside the physical singularity. They allow to remove the non-physical singularity at $r = r_{\rm Schw}$ and provide new insights on the interior solution, on which we will return later.

Let us consider the following coordinate transformation:

$$\begin{aligned} u &= \left(\frac{r}{r_{\rm Schw}} - 1\right)^{1/2} e^{\frac{r}{2r_{\rm Schw}}} \cosh\left(\frac{ct}{2r_{\rm Schw}}\right), \\ v &= \left(\frac{r}{r_{\rm Schw}} - 1\right)^{1/2} e^{\frac{r}{2r_{\rm Schw}}} \sinh\left(\frac{ct}{2r_{\rm Schw}}\right), \\ & \qquad\qquad \text{if } \ r > r_{\rm Schw}, \end{aligned} \tag{176}$$

and

$$\begin{aligned} u &= \left(1 - \frac{r}{r_{\rm Schw}}\right)^{1/2} e^{\frac{r}{2r_{\rm Schw}}} \sinh\left(\frac{ct}{2r_{\rm Schw}}\right), \\ v &= \left(1 - \frac{r}{r_{\rm Schw}}\right)^{1/2} e^{\frac{r}{2r_{\rm Schw}}} \cosh\left(\frac{ct}{2r_{\rm Schw}}\right), \\ & \qquad\qquad \text{if } \ r < r_{\rm Schw}. \end{aligned} \tag{177}$$

The line element in the Kruskal-Szekeres coordinates is completely regular, except at $r = 0$:

$$ds^2 = \frac{4r_{\rm Schw}^3}{r} e^{\frac{r}{r_{\rm Schw}}} \left(dv^2 - du^2\right) - r^2 d\Omega^2. \tag{178}$$

The curves at $r =$ constant are hyperbolic and satisfy:

$$u^2 - v^2 = \left(\frac{r}{r_{\rm Schw}} - 1\right)^{1/2} e^{\frac{r}{r_{\rm Schw}}}, \tag{179}$$

whereas the curves at $t =$ constant are straight lines that pass through the origin:

$$\begin{aligned} \frac{u}{v} &= \tanh\frac{ct}{2r_{\rm Schw}}, \quad r < r_{\rm Schw}, \\ \frac{u}{v} &= \coth\frac{ct}{2r_{\rm Schw}}, \quad r > r_{\rm Schw}. \end{aligned} \tag{180}$$

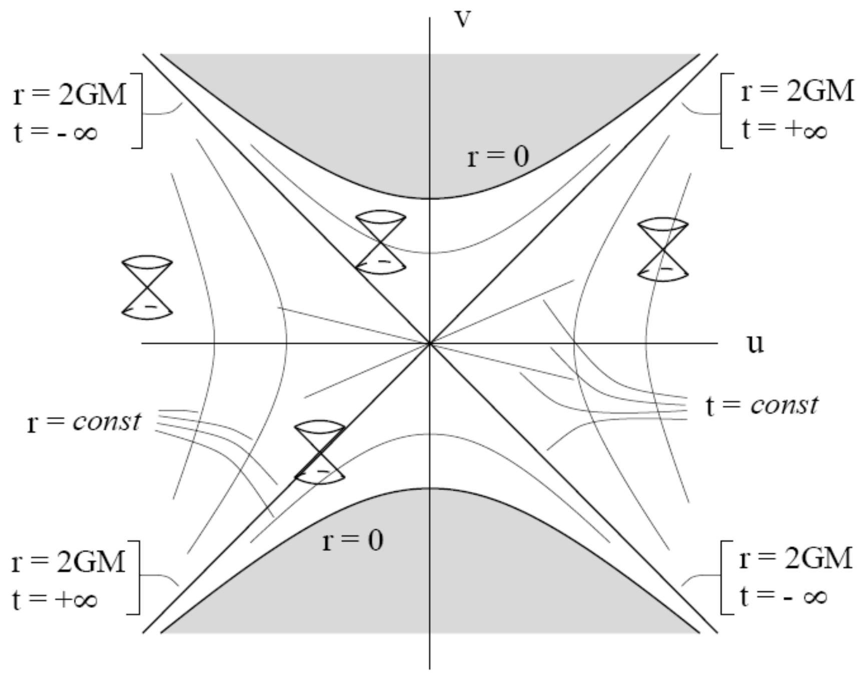

Figure 9. The Schwarzschild metric in Kruskal-Szekeres coordinates ($c = 1$).

In Figure 9 we show the Schwarzschild space-time in Kruskal-Szekeres coordinates. Each hyperbola represents a set of events of constant radius in Schwarzschild coordinates. A radial worldline of a photon in this diagram ($ds = 0$) is represented by a straight line forming an angle of $\pm 45°$with the u axis. A time-like trajectory has always a slope larger than that of 45°; and a space-like one, a smaller slope. A particle falling into the black hole crosses the line at 45° and reaches the future singularity at $r = 0$. For an external observer this occurs in an infinite time. The Kruskal-Szekeres coordinates have the useful feature that outgoing null geodesics are given by $u =$ constant, whereas ingoing null geodesics are given by $v =$ constant. Furthermore, the (future and past) event horizon(s) are given by the equation $uv = 0$, and the curvature singularity is given by the equation $uv = 1$.

A closely related diagram is the so-called Penrose or Penrose-Carter diagram. This is a two-dimensional diagram that captures the causal relations between different points in space-time. It is an extension of a Minkowski diagram where the vertical dimension represents time, and the horizontal dimension represents space, and slanted lines at an angle of 45° correspond to light rays. The biggest difference with a Minkowski diagram (light cone) is that, locally, the metric on a Penrose diagram is conformally equivalent[9] to the actual met-

[9]Two geometries are conformally equivalent if there exists a conformal transformation (an angle-preserving transformation) that maps one geometry to the other one. More generally, two

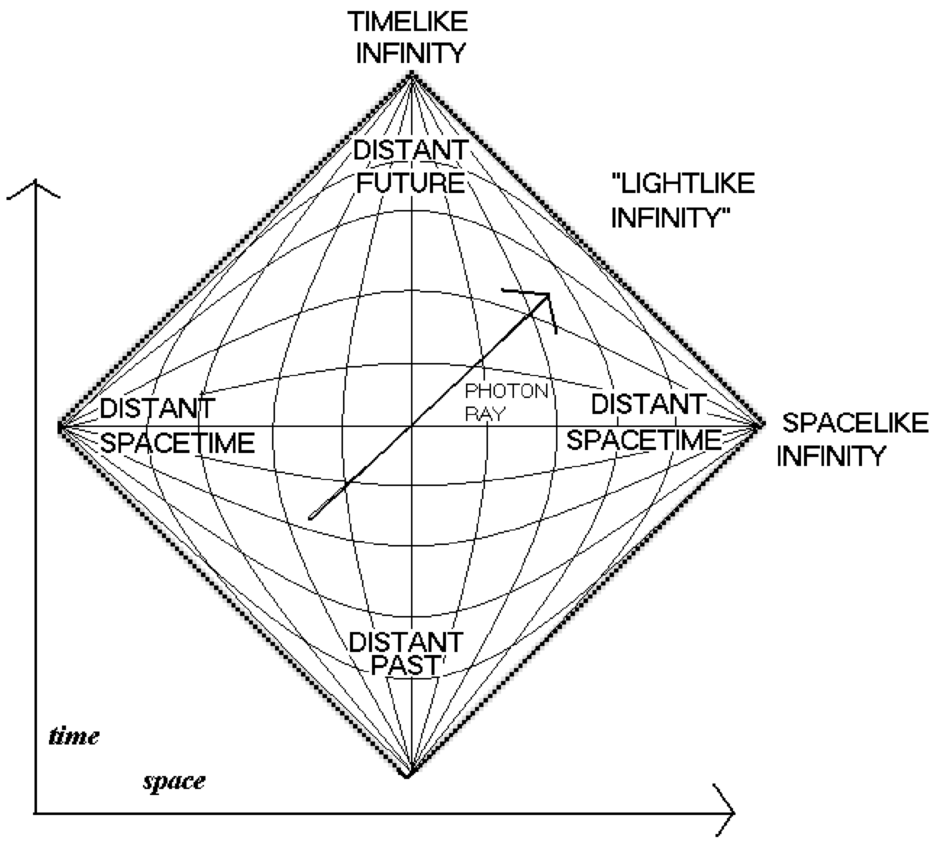

Figure 10. Penrose diagram of a Minkowskian space-time.

ric in space-time. The conformal factor is chosen such that the entire infinite space-time is transformed into a Penrose diagram of finite size. For spherically symmetric space-times, every point in the diagram corresponds to a 2-sphere. In Figure 10 we show a Penrose diagram of a Minkowskian space-time.

This type of diagrams can be applied to Schwarzschild black holes. The result is shown in Figure 11. The trajectory represents a particle that goes from some point in our universe into the black hole, ending in the singularity. Notice that there is a mirror extension, also present in the Kruskal-Szekeres diagram, representing a white hole and a parallel, but inaccessible universe. A white hole presents a naked singularity. These type of extensions of solutions of Einstein's field equations will be discussed later.

Now, we turn to axially symmetric (rotating) solutions of the field equations.

Riemannian metrics on a manifold M are conformally equivalent if one is obtained from the other through multiplication by a function on M.

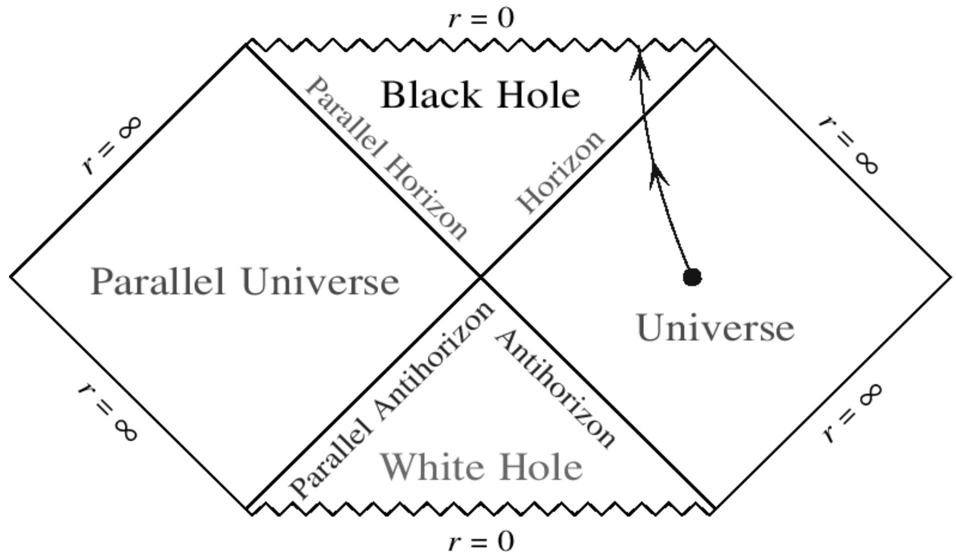

Figure 11. Penrose diagram of a Schwarzschild black hole.

17. Kerr black holes

A Schwarzschild black hole does not rotate. The solution of the field equations (35) for a rotating body of mass M and angular momentum per unit mass a was found by Roy Kerr (1963):

$$\begin{aligned} ds^2 &= g_{tt}dt^2 + 2g_{t\phi}dtd\phi - g_{\phi\phi}d\phi^2 - \Sigma\Delta^{-1}dr^2 - \Sigma d\theta^2 && (181)\\ g_{tt} &= (c^2 - 2GMr\Sigma^{-1}) && (182)\\ g_{t\phi} &= 2GMac^{-2}\Sigma^{-1}r\sin^2\theta && (183)\\ g_{\phi\phi} &= [(r^2 + a^2c^{-2})^2 - a^2c^{-2}\Delta\sin^2\theta]\Sigma^{-1}\sin^2\theta && (184)\\ \Sigma &\equiv r^2 + a^2c^{-2}\cos^2\theta && (185)\\ \Delta &\equiv r^2 - 2GMc^{-2}r + a^2c^{-2}. && (186)\end{aligned}$$

This is the Kerr metric in Boyer-Lindquist coordinates $(t,\ r,\ \theta,\ \phi)$, which reduces to Schwarzschild metric for $a = 0$. In Boyer-Lindquist coordinates the metric is approximately Lorentzian at infinity (i.e. we have a Minkowski space-time in the usual coordinates of Special Relativity).

Note that the element $g_{t\phi}$ no longer vanishes. Even at infinity this element remains (hence we wrote *approximately* Lorentzian above). The Kerr parameter ac^{-1} has dimensions of length. The larger the ratio of this scale to GMc^{-2} (the *spin parameter* $a_* \equiv ac/GM$), the more aspherical the metric. Schwarzschild's black hole is the special case of Kerr's for $a = 0$. Notice that, with the adopted conventions, the angular momentum J is related to the parameter a by:

$$J = Ma. \qquad (187)$$

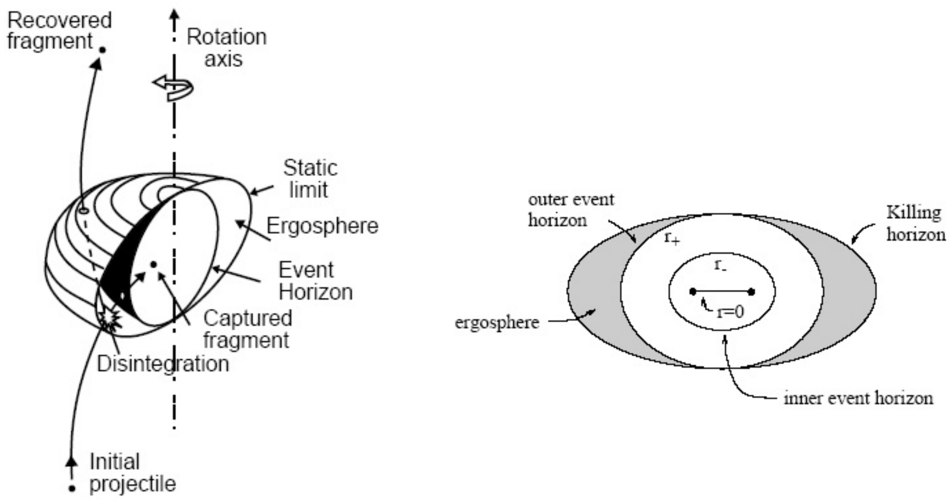

Figure 12. Left: A rotating black hole and the Penrose process. Adapted from J-P. Luminet (1998). *Right*: Sketch of the interior of a Kerr black hole.

Just as the Schwarzschild solution is the unique static vacuum solution of Eqs. (35) (a result called Israel's theorem), the Kerr metric is the unique stationary axisymmetric vacuum solution (Carter-Robinson theorem).

The horizon, the surface which cannot be crossed outward, is determined by the condition $g_{rr} \to \infty$ ($\Delta = 0$). It lies at $r = r_{\rm h}$ where

$$r_{\rm h} \equiv GMc^{-2} + [(GMc^{-2})^2 - a^2c^{-2}]^{1/2}. \tag{188}$$

Indeed, the track $r = r_{\rm h}$, $\theta =$ constant with $d\phi/d\tau = a(r_{\rm h}^2 + a^2)^{-1}\, dt/d\tau$ has $d\tau = 0$ (it represents a photon circling azimuthaly *on* the horizon, as opposed to hovering at it). Hence the surface $r = r_{\rm h}$ is tangent to the local light cone. Because of the square root in Eq. (188), the horizon is well defined only for $a_* = ac/GM \leq 1$. An *extreme* (i.e. maximally rotating) Kerr black hole has a spin parameter $a_* = 1$. Notice that for $(GMc^{-2})^2 - a^2c^{-2} > 0$ we have actually two horizons. The second, the *inner* horizon, is located at:

$$r_{\rm h}^{\rm inn} \equiv GMc^{-2} - [(GMc^{-2})^2 - a^2c^{-2}]^{1/2}. \tag{189}$$

This horizon is not seen by an external observer, but it hides the singularity to any observer that has already crossed r_h and is separated from the rest of the universe. For $a = 0$, $r_{\rm h}^{\rm inn} = 0$ and $r_{\rm h} = r_{\rm Schw}$. The case $(GMc^{-2})^2 - a^2c^{-2} < 0$ corresponds to no horizons and it is thought to be unphysical.

A study of the orbits around a Kerr black hole is beyond the limits of the present text (the reader is referred to Frolov and Novikov 1998), but we will mention several interesting features. One is that if a particle initially falls radially with no angular momentum from infinity to the black hole, it gains angular motion during the infall. The angular velocity as seen from a distant observer is:

$$\Omega(r,\ \theta) = \frac{d\phi}{dt} = \frac{(2GM/c^2)ar}{(r^2 + a^2c^{-2})^2 - a^2c^{-2}\Delta \sin^2\theta}. \tag{190}$$

Notice that the particle will acquire angular velocity in the direction of the spin of the black hole. As the black hole is approached, the particle will find an increasing tendency to get carried away in the same sense in which the black hole is rotating. To keep the particle stationary respect the distant stars, it will be necessary to apply a force against this tendency. The closer the particle will be to the black hole, the stronger the force. At a point $r_{\rm e}$ it becomes impossible to counteract the rotational sweeping force. The particle is in a kind of space-time maelstrom. The surface determined by $r_{\rm e}$ is the *static limit*: from there in, you cannot avoid to rotate. Space-time is rotating here in such a way that you cannot do anything in order to not co-rotate. You can still escape from the black hole, since the outer event horizon has not been crossed, but rotation is inescapable. The region between the static limit and the event horizon is called the *ergosphere*. The ergosphere is not spherical but its shape changes with the latitude θ. It can be determined through the condition $g_{tt} = 0$. If we consider a stationary particle, r = constant, θ = constant, and ϕ = constant. Then:

$$c^2 = g_{tt} \left(\frac{dt}{d\tau}\right)^2. \tag{191}$$

When $g_{tt} \leq 0$ this condition cannot be fulfilled, and hence a massive particle cannot be stationary inside the surface defined by $g_{tt} = 0$. For photons, since $ds = cd\tau = 0$, the condition is satisfied at the surface. Solving $g_{tt} = 0$ we obtain the shape of the ergosphere:

$$r_{\rm e} = \frac{GM}{c^2} + \frac{1}{c^2}\left(G^2M^2 - a^2c^2\cos^2\theta\right)^{1/2}. \tag{192}$$

The static limit lies outside the horizon except at the poles where both surfaces coincide. The phenomenon of "frame dragging"' is common to all axially symmetric metrics with $d_{t\phi} \neq 0$.

Roger Penrose (1969) suggested that a projectile thrown from outside into the ergosphere begins to rotate acquiring more rotational energy than it originally had. Then the projectile can break up into two pieces, one of which will fall into the black hole, whereas the other can go out of the ergosphere. The piece coming out will then have more energy than the original projectile. In this way, we can extract energy from a rotating black hole. In Fig. 12 we illustrate the situation and show the static limit, the ergosphere and the outer/inner horizons of a Kerr black hole.

The innermost marginally stable circular orbit $r_{\rm ms}$ around a extreme rotating black hole ($ac^{-1} = GM/c^2$) is given by (Raine and Thomas 2005):

$$\left(\frac{r_{\rm ms}}{GM/c^2}\right)^2 - 6\left(\frac{r_{\rm ms}}{GM/c^2}\right) \pm 8\left(\frac{r_{\rm ms}}{GM/c^2}\right)^{1/2} - 3 = 0. \tag{193}$$

For the + sign this is satisfied by $r_{\rm ms} = GM/c^2$, whereas for the − sign the solution is $r_{\rm ms} = 9GM/c^2$. The first case corresponds to a co-rotating particle and the second one to a counter-rotating particle. The energy of the co-rotating particle in the innermost orbit is $1/\sqrt{3}$ (units of mc^2). The binding energy of a particle in an orbit is the difference between the orbital energy and its energy at infinity. This means a binding energy of 42% of the rest energy at infinity!. For the counter-rotating particle, the binding energy is 3.8 %, smaller than for a Schwazschild black hole.

Pseudo-Newtonian potentials for black holes

The full effective general relativistic potential for particle orbits around a Kerr black hole is quite complex. Instead, pseudo-Newtonian potentials are used. The first of such potentials, derived by Bohdam Paczyński and used by first time by Paczyński & Wiita (1980), for a non-rotating black hole with mass M, is:

$$\Phi = -\frac{GM}{r - 2r_{\rm g}}, \tag{194}$$

where as before $r_{\rm g} = GM/c^2$ is the gravitational radius. With this potential one can use Newtonian theory and obtain the same behavior of the Keplerian circular orbits of free particles as in the exact theory: orbits with $r < 9r_{\rm g}$ are unstable, and orbits with $r < 6r_{\rm g}$ are unbound. However, velocities of material particles obtained with the potential (194) are not accurate, since special relativistic effects are not included (Abramowicz et al. 1996). The velocity $v_{\rm p-N}$ calculated with the pseudo-Newtonian potential should be replaced by the corrected velocity $v_{\rm p-N}^{\rm corr}$ such that

$$v_{\rm p-N} = v_{\rm p-N}^{\rm corr}\, \gamma_{\rm p-N}^{\rm corr}, \quad \gamma_{\rm p-N}^{\rm corr} = \frac{1}{\sqrt{1 - \left(\frac{v_{\rm p-N}^{\rm corr}}{c}\right)^2}}. \tag{195}$$

This re-scaling works amazingly well (see Abramowicz et al. 1996) for comparisons with the actual velocities. The agreement with General Relativity is better than 5%.

For the Kerr black hole, a pseudo-Newtonian potential was found by Semerák & Karas (1999). It can be found in the expression (19) of their paper. However, the use of this potential is almost as complicated as dealing with the full effective potential of the Kerr metric in General Relativity.

18. Other black holes

18.1. Reissner-Nordstrøm black holes

The Reissner-Nordstrøm metric is a spherically symmetric solution of Eqs. (35). However, it is not a vacuum solution, since the source has an electric charge Q, and hence there is an electromagnetic field. The energy-momentum tensor of this field is:

$$T_{\mu\nu} = -\mu_0^{-1}(F_{\mu\rho}F_{\nu^\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}), \tag{196}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor and A_μ is the electromagnetic 4-potential. Outside the charged object the 4-current j^μ is zero, so the Maxwell equations are:

$$F^{\mu\nu};_\mu = 0, \tag{197}$$

$$F_{\mu\nu};_\sigma + F_{\sigma\mu};_\nu + F_{\nu\sigma};_\mu = 0. \tag{198}$$

The Einstein and Maxwell equations are coupled since $F^{\mu\nu}$ enters into the gravitational field equations through the energy-momentum tensor and the metric

$g_{\mu\nu}$ enters into the electromagnetic equations through the covariant derivative. Because of the symmetry constraints we can write:

$$[A^{\mu}] = \left(\frac{\varphi(r)}{c^2},\ a(r),\ 0,\ 0\right), \tag{199}$$

where $\varphi(r)$ is the electrostatic potential, and $a(r)$ is the radial component of the 3-vector potential as $r \longrightarrow \infty$.

The solution for the metric is given by

$$ds^2 = \Delta c^2 dt^2 - \Delta^{-1} dr^2 - r^2 d\Omega^2 \ , \tag{200}$$

where

$$\Delta = 1 - \frac{2GM/c^2}{r} + \frac{q^2}{r^2}. \tag{201}$$

In this expression, M is once again interpreted as the mass of the hole and

$$q = \frac{GQ^2}{4\pi\epsilon_0 c^4} \tag{202}$$

is related to the total electric charge Q.

The metric has a coordinate singularity at $\Delta = 0$, in such a way that:

$$r_{\pm} = r_{\rm g} \pm (r_{\rm g}^2 - q^2)^{1/2}. \tag{203}$$

Here, $r_{\rm g} = GM/c^2$ is the gravitational radius. For $r_{\rm g} = q$, we have an *extreme* Reissner-Nordstrøm black hole with a unique horizon at $r = r_{\rm g}$. Notice that a Reissner-Nordstrøm black hole can be more compact than a Scharzschild black hole of the same mass. For the case $r_{\rm g}^2 > q^2$, both $r_{\pm}$ are real and there are two horizons as in the Kerr solution. Finally, in the case $r_{\rm g}^2 < q^2$ both $r_{\pm}$ are imaginary there is no coordinate singularities, no horizon hides the intrinsic singularity at $r = 0$. It is thought, however, that naked singularities do not exist in nature (see Section 24. below).

18.2. Kerr-Newman black holes

The Kerr-Newman metric of a charged spinning black hole is the most general black hole solution. It was found by Ezra “Ted” Newman in 1965 (Newman et al. 1965). This metric can be obtained from the Kerr metric (181) in Boyer-Lindquist coordinates by the replacement:

$$\frac{2GM}{c^2} r \longrightarrow \frac{2GM}{c^2} r - a^2 c^{-2} - q^2,$$

where q is related to the charge Q by Eq. (202).

The full expression reads:

$$\begin{aligned}
ds^2 &= g_{tt} dt^2 + 2 g_{t\phi} dt d\phi - g_{\phi\phi} d\phi^2 - \Sigma \Delta^{-1} dr^2 - \Sigma d\theta^2 && (204)\\
g_{tt} &= (c^2 - 2GMr\Sigma^{-2}) && (205)\\
g_{t\phi} &= 2GMac^{-2}\Sigma^{-1} r \sin^2\theta && (206)\\
g_{\phi\phi} &= [(r^2 + a^2c^{-2})^2 - a^2c^{-2}\Delta \sin^2\theta]\Sigma^{-1}\sin^2\theta && (207)\\
\Sigma &\equiv r^2 + a^2c^{-2}\cos^2\theta && (208)\\
\Delta &\equiv r^2 - 2GMc^{-2}r + a^2c^{-2} + q^2 \equiv (r - r_{\rm h}^{\rm out})(r - r_{\rm h}^{\rm inn}), && (209)
\end{aligned}$$

where all the symbols have the same meaning as in the Kerr metric and the outer horizon is located at:

$$r_{\rm h}^{\rm out} = GMc^{-2} + [(GMc^{-2})^2 - a^2c^{-2} - q^2]^{1/2}. \qquad (210)$$

The inner horizon is located at:

$$r_{\rm h}^{\rm inn} = GMc^{-2} - [(GMc^{-2})^2 - a^2c^{-2} - q^2]^{1/2}. \qquad (211)$$

The Kerr-Newman solution is a non-vacuum solution, as the Reissner-Nordstrøm is. It shares with Kerr and Reissner-Nordstrøm solutions the existence of two horizons, and as the Kerr solution it presents an ergosphere. At a latitude θ, the radial coordinate for the ergosphere is:

$$r_{\rm e} = GMc^{-2} - [(GMc^{-2})^2 - a^2c^{-2}\sin^2\theta - q^2]^{1/2}. \qquad (212)$$

Like the Kerr metric for an uncharged rotating mass, the Kerr-Newman interior solution exists mathematically but is probably not representative of the actual metric of a physically realistic rotating black hole due to stability issues. The surface area of the horizon is:

$$A = 4\pi(r_{\rm h}^{\rm out\ 2} + a^2c^{-2}). \qquad (213)$$

The Kerr-Newman metric represents the simplest stationary, axisymmetric, asymptotically flat solution of Einstein's equations in the presence of an electromagnetic field in four dimensions. It is sometimes referred to as an "electrovaccum" solution of Einstein's equations. Any Kerr-Newman source has its rotation axis aligned with its magnetic axis (Punsly 1998a). Thus, a Kerr-Newman source is different from commonly observed astronomical bodies, for which there is a substantial angle between the rotation axis and the magnetic moment.

Since the electric field cannot remain static in the ergosphere, a magnetic field is generated as seen by an observer outside the static limit. This is illustrated in Figure 13.

Pekeris & Frankowski (1987) have calculated the interior electromagnetic field of the Kerr-Newman source, i.e., the ring singularity. The electric and magnetic fields are shown in Figures 14 and 15, in a $(\lambda,\ z)$-plane, with $\lambda = (x^2+y^2)^{1/2}$. The general features of the magnetic field are that at distances much larger than ac^{-1} it resembles closely a dipole field, with a dipolar magnetic moment $\mu_{\rm d} = Qac^{-1}$. On the disc of radius ac^{-1} the z-component of the field vanishes, in contrast with the interior of Minkowskian ring-current models. The electric field for a positive charge distribution is attractive for positive charges toward the interior disc. At the ring there is a charge singularity and at large distances the field corresponds to that of a point like charge Q.

Charged black holes might be a natural result from charge separation during the gravitational collapse of a star. It is thought that an astrophysical charged object would discharge quickly by accretion of charges of opposite sign. However, there remains the possibility that the charge separation could lead to a configuration where the black hole has a charge and a superconducting ring around it would have the same but opposite charge, in such a way the whole system seen from infinity is neutral. In such a case a Kerr-Newman black hole might survive

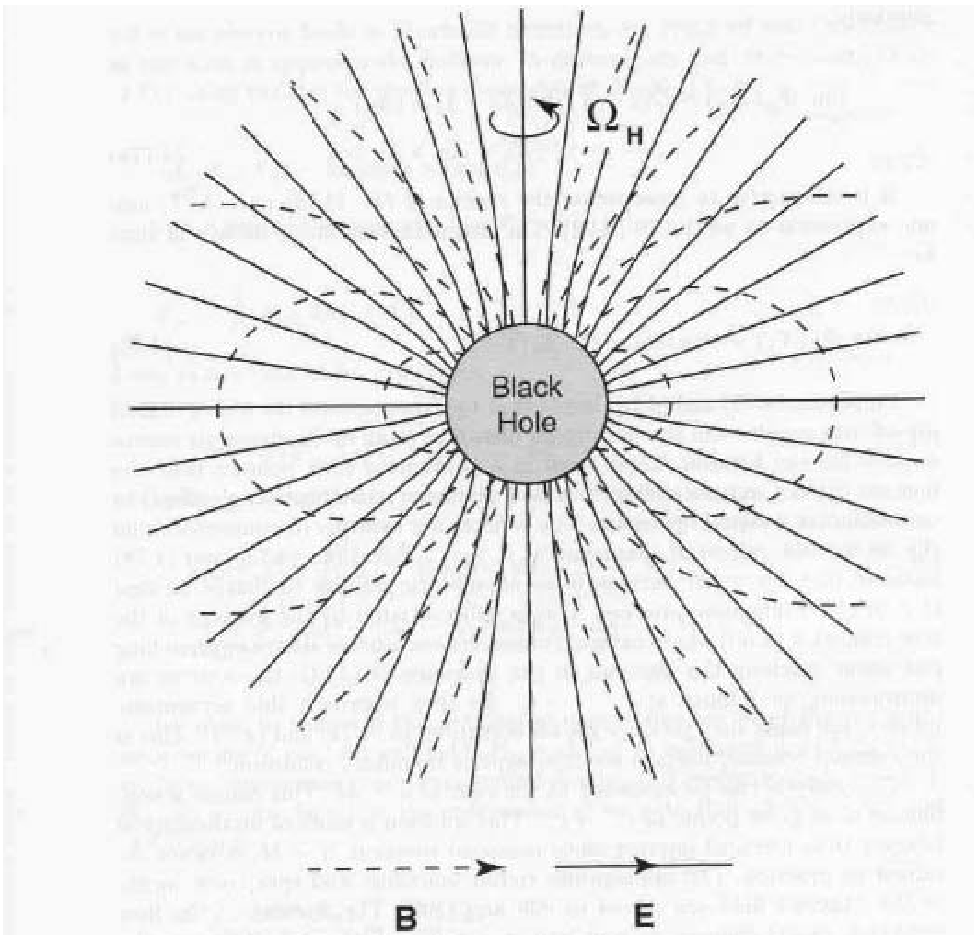

Figure 13. The electric and magnetic field lines of a Kerr-Newman black hole. Adapted from Punsly (2001) and Hanni & Ruffini (1973).

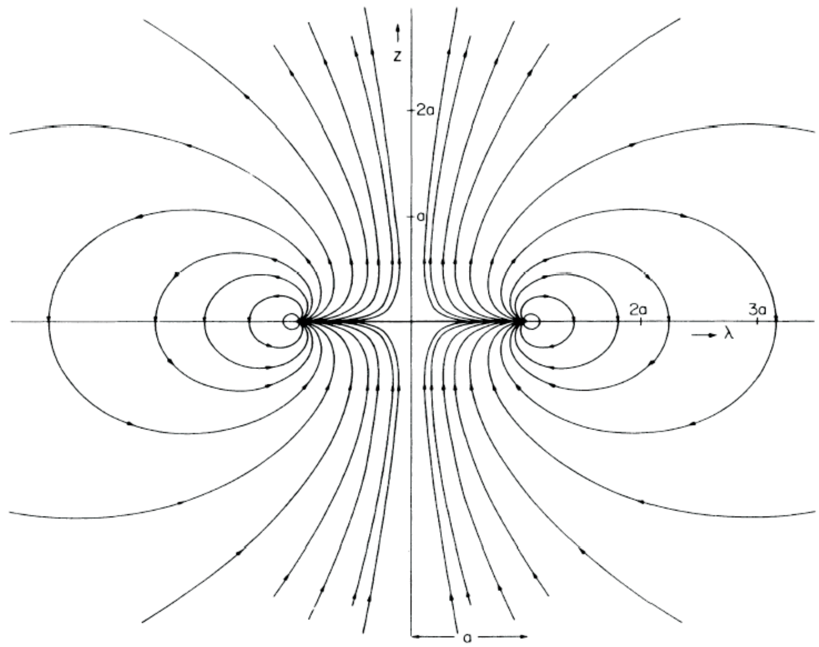

Figure 14. Magnetic field of a Kerr-Newman source. See text for units. From Pekeris & Frankowski (1987).

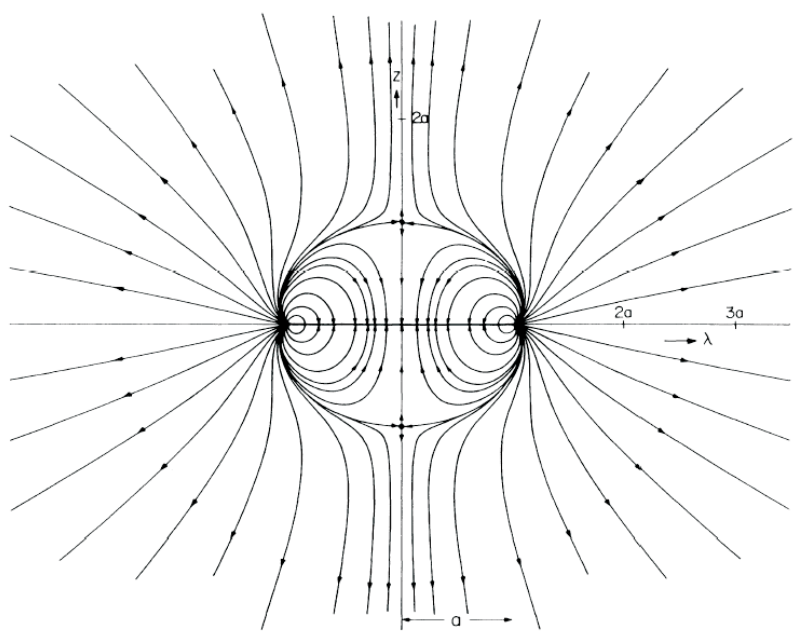

Figure 15. Electric field of a Kerr-Newman source. See text for units. From Pekeris & Frankowski (1987).

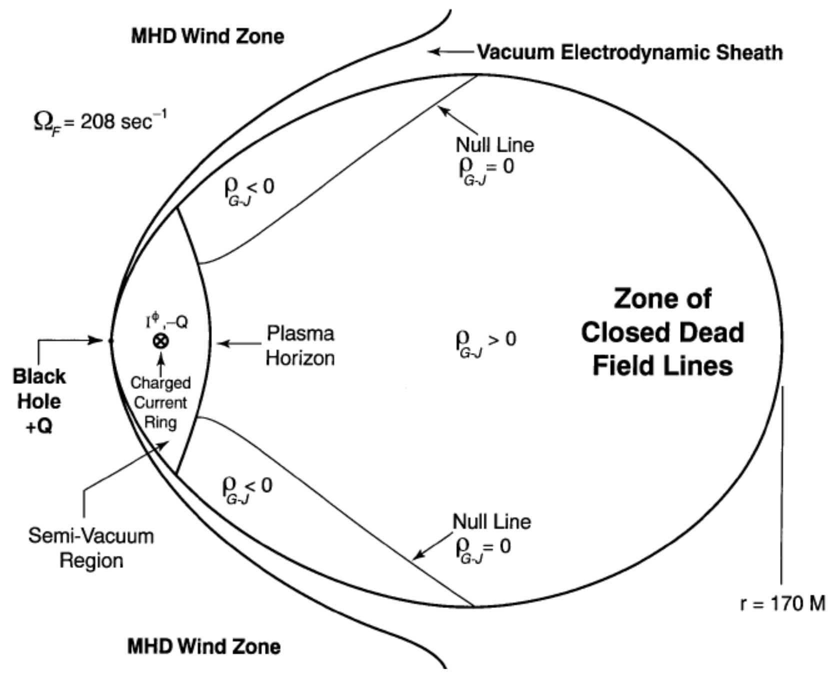

Figure 16. Charged and rotating black hole magnetosphere. The black hole has charge $+Q$ whereas the current ring circulating around it has opposite charge. The figure shows (units $G = c = 1$) the region of closed lines determined by the light cylinder, the open lines that drive a magneto-hydrodynamical wind, and vacuum region in between. From Punsly (1998a).

for some time, depending on the environment. For further details, the reader is referred to the highly technical book by Brian Punsly (2001) and related articles (Punsly 1998a, b, and Punsly et al 2000). In Figures 16, and 17 the magnetic field around a Kerr-Newman black hole surrounded by charged current ring are shown. The opposite charged black hole and ring are the minimum energy configuration for the system black hole plus magnetosphere. Since the system is neutral from the infinity, it discharges slowly and can survive for a few thousand years. During this period, the source can be active, through the capture of free electrons from the environment and the production of gamma rays by inverse Compton up-scattering of synchrotron photons produced by electrons accelerated in the polar gap of the hole. In Figure 18 we show the corresponding spectral energy distribution obtained by Punsly et al. (2000) for such a configuration of Kerr-Newman black hole magnetosphere.

Einstein-Maxwell equations

In order to determine the gravitational and electromagnetic fields over a region of a space-time we have to solve the Einstein-Maxwell equations:

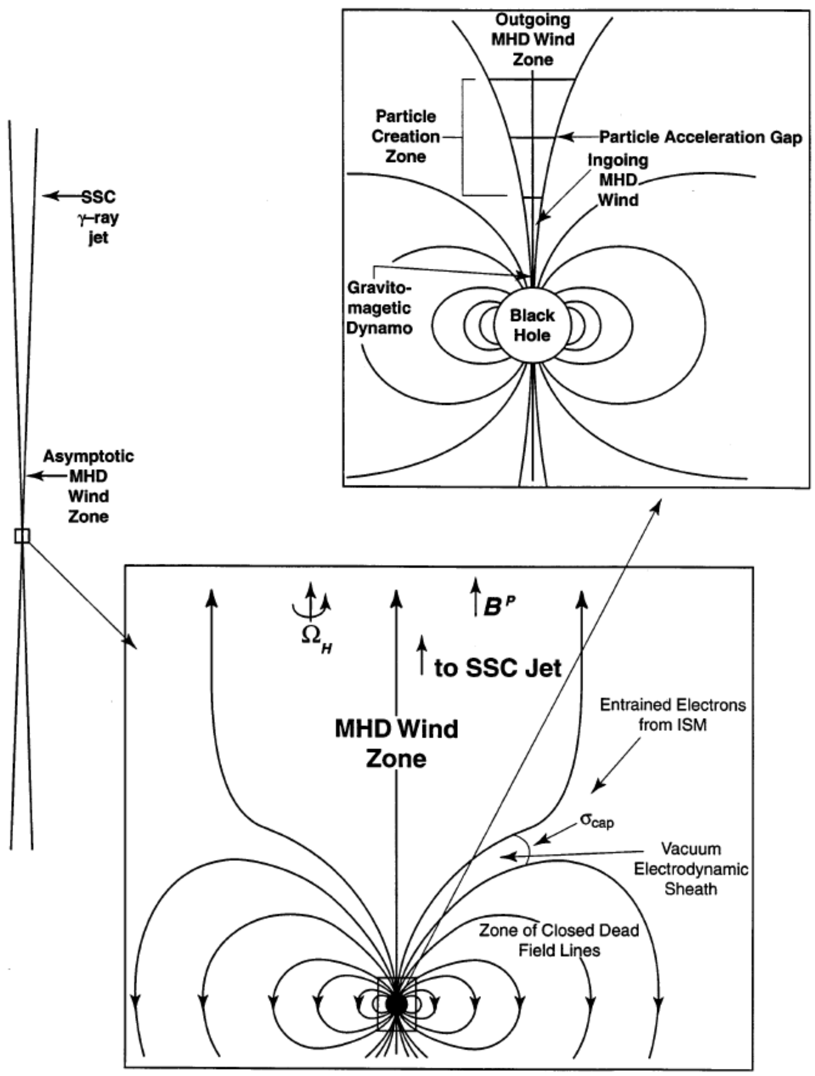

Figure 17. Three different scales of the Kerr-Newman black hole model developed by Brian Punsly. From Punsly (1998a).

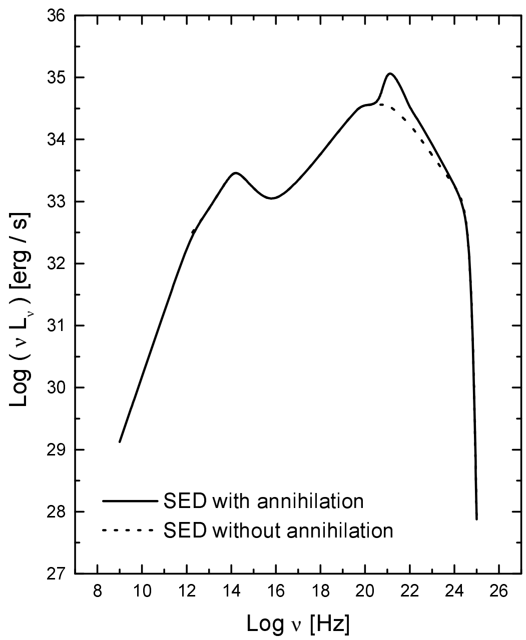

Figure 18. The spectral energy distribution resulting from a Kerr-Newman black hole slowly accreting from the interstellar medium. Adapted from Punsly et al. (2000).

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}=-\frac{8\pi G}{c^4}\left(T_{\mu\nu}+E_{\mu\nu}\right), \tag{214}$$

$$\frac{4\pi}{c}E^{\mu}_{\nu}=-F^{\mu\rho}F_{\rho\nu}+\frac{1}{4}\delta^{\mu}_{\nu}F^{\sigma\lambda}F_{\sigma\lambda}, \tag{215}$$

$$F_{\mu\nu}=A_{\mu};_{\nu}-A_{\nu};_{\mu}, \tag{216}$$

$$F^{\nu}_{\mu};_{\nu}=\frac{4\pi}{c}J_{\mu}. \tag{217}$$

Here $T_{\mu\nu}$ and $E_{\mu\nu}$ are the energy-momentum tensors of matter and electromagnetic fields, $F_{\mu\nu}$ and J_{μ} are the electromagnetic field and current density, A_{μ} is the 4-dimensional potential and Λ is the cosmological constant.

The solution of this system of equations is non-trivial since they are coupled. The electromagnetic field is a source of the gravitational field and this field enters into the electromagnetic equations through the covariant derivatives indicated by the semi-colons. For an exact and relevant solution of the problem see Manko & Sibgatullin (1992).

18.3. Born-Infeld black holes

Born and Infeld (1934) proposed a nonlinear theory of electrodynamics to avoid the singularities associated with charged point particles in Maxwell theory. Almost immediately, Hoffmann (1935) coupled general relativity with Born-Infeld electrodynamics to obtain a spherically symmetric solution representing the gravitational field of a charged object. This solution, forgotten during decades, can represent a charged black hole in nonlinear electrodynamics. In Born–Infeld electrodynamics the trajectories of photons in curved spacetimes are not null geodesics of the background metric. Instead, they follow null geodesics of an effective geometry determined by the nonlinearities of the electromagnetic field.

The action of Einstein gravity coupled to Born–Infeld electrodynamics has the form (in this section we adopt, for simplicity in the notation, $c=G=4\pi\epsilon_0=(4\pi)^{-1}\mu_0=0$):

$$S=\int dx^4\sqrt{-g}\left(\frac{R}{16\pi}+L_{\rm BI}\right), \tag{218}$$

with

$$L_{\rm BI}=\frac{1}{4\pi b^2}\left(1-\sqrt{1+\frac{1}{2}F_{\sigma\nu}F^{\sigma\nu}b^2-\frac{1}{4}\tilde{F}_{\sigma\nu}F^{\sigma\nu}b^4}\right), \tag{219}$$

where g is the determinant of the metric tensor, R is the scalar of curvature, $F_{\sigma\nu}=\partial_\sigma A_\nu-\partial_\nu A_\sigma$ is the electromagnetic tensor, $\tilde{F}_{\sigma\nu}=\frac{1}{2}\sqrt{-g}\,\varepsilon_{\alpha\beta\sigma\nu}F^{\alpha\beta}$ is the dual of $F_{\sigma\nu}$ (with $\varepsilon_{\alpha\beta\sigma\nu}$ the Levi–Civita symbol) and b is a parameter that indicates how much Born–Infeld and Maxwell electrodynamics differ. For $b\to 0$ the Einstein–Maxwell action is recovered. The maximal possible value of the electric field in this theory is b, and the self-energy of point charges is finite. The field equations can be obtained by varying the action with respect to the metric $g_{\sigma\nu}$ and the electromagnetic potential A_ν.

We can write $L_{\rm BI}$ in terms of the electric and magnetic fields:

$$L_{\rm BI} = \frac{b^2}{4\pi}\left[1-\sqrt{1-\frac{B^2-E^2}{b^2}-\frac{(\vec{E}\vec{B})^2}{b^4}}\right]. \tag{220}$$

The Larangian depends non-linearly of the electromagnetic invariants:

$$F = \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} = \frac{1}{2}(B^2-E^2), \tag{221}$$

$$\tilde{G} = \frac{1}{4}F_{\alpha\beta}\tilde{F}^{\alpha\beta} = -\bar{B}\cdot\bar{E}. \tag{222}$$

Introducing the Hamiltonian formalism:

$$P^{\alpha\beta} = 2\frac{\partial L}{\partial F_{\alpha\beta}} = \frac{\partial L}{\partial F}F^{\alpha\beta} + \frac{\partial L}{\partial \tilde{G}}\tilde{F}^{\alpha\beta}, \tag{223}$$

$$H = \frac{1}{2}P^{\alpha\beta}F_{\alpha\beta} - L(F,\tilde{G}^2), \tag{224}$$

and adopting the notation

$$P = \frac{1}{4}P_{\alpha\beta}P^{\alpha\beta}, \tag{225}$$

$$\tilde{Q} = \frac{1}{4}P_{\alpha\beta}\tilde{P}^{\alpha\beta}, \tag{226}$$

we can express $F^{\alpha\beta}$ as a function of $P^{\alpha\beta}$, P, and $\tilde{Q}$:

$$F^{\alpha\beta} = 2\frac{\partial H}{\partial P_{\alpha\beta}} = \frac{\partial H}{\partial P}P^{\alpha\beta} + \frac{\partial H}{\partial \tilde{Q}}\tilde{P}^{\alpha\beta}. \tag{227}$$

The Hamiltonian equations in the P and $\tilde{Q}$ formalism can be written as:

$$\left(\frac{\partial H}{\partial P}\tilde{P}^{\alpha\beta} + \frac{\partial H}{\partial \tilde{Q}}P^{\alpha\beta}\right)_{,\beta} = 0. \tag{228}$$

.

The couple Einstein-Born-Infeld equations are:

$$4\pi T_{\mu\nu} = \frac{\partial H}{\partial P}P_{\mu\alpha}P^{\alpha}_{\nu} - g_{\mu\nu}\left(2P\frac{\partial H}{\partial P} + \tilde{Q}\frac{\partial H}{\partial \tilde{Q}} - H\right), \tag{229}$$

$$R = 8\left(P\frac{\partial H}{\partial P} + \tilde{Q}\frac{\partial H}{\partial \tilde{Q}} - H\right). \tag{230}$$

The field equations have spherically symmetric black hole solutions given by

$$ds^2 = \psi(r)dt^2 - \psi(r)^{-1}dr^2 - r^2 d\Omega^2, \tag{231}$$

with

$$\psi(r) = 1 - \frac{2M}{r} + \frac{2}{b^2 r}\int_r^{\infty}\left(\sqrt{x^4+b^2Q^2} - x^2\right)dx, \tag{232}$$

$$D(r) = \frac{Q_E}{r^2}, \tag{233}$$

$$B(r) = Q_M \sin\theta, \tag{234}$$

where M is the mass, $Q^2 = Q_E^2 + Q_M^2$ is the sum of the squares of the electric Q_E and magnetic Q_M charges, $B(r)$ and $D(r)$ are the magnetic and the electric inductions in the local orthonormal frame. In the limit $b \to 0$, the Reissner–Nordström metric is obtained. The metric (231) is also asymptotically Reissner–Norsdtröm for large values of r. With the units adopted above, M, Q and b have dimensions of length. The metric function $\psi(r)$ can be expressed in the form

$$\psi(r) = 1 - \frac{2M}{r} + \frac{2}{3b^2}\left\{r^2 - \sqrt{r^4 + b^2Q^2} + \frac{\sqrt{|bQ|^3}}{r} F\left[\arccos\left(\frac{r^2 - |bQ|}{r^2 + |bQ|}\right), \frac{\sqrt{2}}{2}\right]\right\}, \tag{235}$$

where $F(\gamma, k)$ is the elliptic integral of the first kind[10]. As in Schwarzschild and Reissner–Norsdtröm cases, the metric (231) has a singularity at $r = 0$.

The zeros of $\psi(r)$ determine the position of the horizons, which have to be obtained numerically. For a given value of b, when the charge is small, $0 \leq |Q|/M \leq \nu_1$, the function $\psi(r)$ has one zero and there is a regular event horizon. For intermediate values of charge, $\nu_1 < |Q|/M < \nu_2$, $\psi(r)$ has two zeros, so there are, as in the Reissner–Nordström geometry, an inner horizon and an outer regular event horizon. When $|Q|/M = \nu_2$, there is one degenerate horizon. Finally, if the values of charge are large, $|Q|/M > \nu_2$, the function $\psi(r)$ has no zeros and a naked singularity is obtained. The values of $|Q|/M$ where the number of horizons change, $\nu_1 = (9|b|/M)^{1/3}[F(\pi, \sqrt{2}/2)]^{-2/3}$ and ν_2, which should be calculated numerically from the condition $\psi(r_h) = \psi'(r_h) = 0$, are increasing functions of $|b|/M$. In the Reissner–Nordström limit ($b \to 0$) it is easy to see that $\nu_1 = 0$ and $\nu_2 = 1$.

The paths of photons in nonlinear electrodynamics are not null geodesics of the background geometry. Instead, they follow null geodesics of an effective metric generated by the self-interaction of the electromagnetic field, which depends on the particular nonlinear theory considered. In Einstein gravity coupled to Born–Infeld electrodynamics the effective geometry for photons is given by :

$$ds^2_{\text{eff}} = \omega(r)^{1/2}\psi(r)dt^2 - \omega(r)^{1/2}\psi(r)^{-1}dr^2 - \omega(r)^{-1/2}r^2 d\Omega^2, \tag{236}$$

where

$$\omega(r) = 1 + \frac{Q^2b^2}{r^4}. \tag{237}$$

Then, to calculate the deflection angle for photons passing near the black holes, it is necessary to use the effective metric (236) instead of the background metric (231). The horizon structure of the effective metric is the same that of metric (231), but the trajectories of photons are different.

---

[10] $F(\gamma, k) = \int_0^\gamma (1 - k^2 \sin^2\phi)^{-1/2} d\phi = \int_0^{\sin\gamma} [(1 - z^2)(1 - k^2 z^2)]^{-1/2} dz$

18.4. Regular black holes

Solutions of Einstein's field equations representing black holes where the metric is always regular (i.e. free of intrinsic singularities with $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ diverges) can be found for some choices of the equation of state. For instance, Mbonye and Kazanas (2005) have sugested the following equation:

$$p_r(\rho) = \left[\alpha - (\alpha+1)\left(\frac{\rho}{\rho_{\max}}\right)^m\right]\left(\frac{\rho}{\rho_{\max}}\right)^{1/n}\rho. \tag{238}$$

The maximum limiting density $\rho_{\max}$ is concentrated in a region of radius:

$$r_0 = \sqrt{\frac{1}{G\rho_{\max}}}. \tag{239}$$

At low densities $p_r \propto \rho^{1+1/n}$ and the equation reduces to that of a polytrope gas. At high densities, close to $\rho_{\max}$ the equation becomes $p_r = -\rho$ and the system behaves as a gravitational field dominated by a cosmological term in the field equations. The exact values of m, n, and α determine the sound speed in the system. Imposing that the maximum sound speed $c_{\rm s} = (dp/d\rho)^{1/2}$ be finite everywhere, is possible to constrain the free parameters. Adopting $m = 2$ and $n = 1$ the Eq. (238) becomes:

$$p_r(\rho) = \left[\alpha - (\alpha+1)\left(\frac{\rho}{\rho_{\max}}\right)^2\right]\left(\frac{\rho}{\rho_{\max}}\right)\rho. \tag{240}$$

Then, the Schwarzschild solution can be written as:

$$ds^2 = \left(1 - \frac{2GM(r)}{rc^2}\right)e^{\Gamma(r)}c^2dt^2 - \left(1 - \frac{2GM(r)}{rc^2}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{241}$$

where $M(r)$ is the mass enclosed by a 2-sphere of radius r and:

$$\Gamma(r) = \int 8\pi\left[\alpha - (\alpha+1)\left(\frac{\rho}{\rho_{\max}}\right)^2\right]\left(\frac{\rho}{\rho_{\max}}\right)\left(\frac{r}{r - 2GM(r)}\right)\rho dr. \tag{242}$$

The mass within r is

$$M(r) = 4\pi\int_0^r \rho(r')r'^2dr', \tag{243}$$

and the total mass is

$$M = \int_0^\infty M(r)dr = \int_0^\infty \rho(r)r^2dr. \tag{244}$$

Since outside the body $\rho \to 0$, $\Gamma(r) \to 1$ and Eq. (241) becomes Schwarzschild solution for $R_{\mu\nu} = 0$.

When $r \to 0$, $\rho = \rho_{\max}$ and the metric becomes of de Sitter type:

$$ds^2 = \left(1 - \frac{r^2}{r_0^2}\right)c^2dt^2 - \left(1 - \frac{r^2}{r_0^2}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{245}$$

with

$$r_0 = \sqrt{\frac{3}{8\pi G \rho_{\max}}}. \tag{246}$$

There is no singularity at $r = 0$ and the black hole is regular. For $0 \leq r < 1$ has constant positive density $\rho_{\max}$ and negative pressure $p_r = -\rho_{\max}$ and space-time becomes asymptotically de Sitter in the inner most region. It might be speculated that the transition in the equation of state occurs because at very high densities the matter field couples with a scalar field that provides the negative pressure.

Other assumptions for the equation of state can lead to different (but still regular) behavior, like a bouncing colse to $r = 0$ and the development of an expanding closed universe inside the black hole (Frolov, Markov, and Mukhanov 1990). In addition, regular black holes can be found in $f(R)$ gravity for some suitable function of the curvature scalar.

19. Black hole formation

Black holes will form every time that matter and fields are compressed beyond the corresponding Schwarzschild radius. This can occur in a variety of forms, from particle collisions to the implosion of stars or the collapse of dark matter in the early universe. The most common black hole formation mechanism in our Galaxy seems to be gravitational collapse. A normal star is stable as long as the nuclear reactions occurring in its interior provide thermal pressure to support it against gravity. Nuclear burning gradually transforms the stellar core from H to He and in the case of massive stars ($M > 5\ M_{\odot}$) then to C and finally to Fe. The core contracts in the processes, in order to achieve the ignition of each phase of thermonuclear burning.

Finally, the endothermic disintegration of iron-group nuclei, which are those with the tightest bound, precipitates the collapse of the core to a stellar-mass black hole. Stars with masses in the range $20 - 30\ M_{\odot}$ produce black holes with $M > 1.8\ M_{\odot}$. Low-mass black holes ($1.5\ M_{\odot} < M < 1.8\ M_{\odot}$) can result from the collapse of stars of $18 - 20\ M_{\odot}$ along with the ejection of the outer layers of the star by a shock wave in an event known as Type II supernova. A similar event, occurring in stars with $10 - 18\ M_{\odot}$ leaves behind a neutron star. Very massive stars with high spin likely end producing a gamma-ray burst and a very massive ($M > 10\ M_{\odot}$) black hole. The binary stellar systems have a different evolution. The interested reader can find an comprehensive review in Brown et al. (2000).

In Figure 19 we show the Eddignton-Finkelstein diagram of the gravitational collapse of a star. Once the null surface of the light cones points along the time axis the black hole is formed: light rays will never be again able to escape to the outer universe. The different paths that can lead to a stellar mass black hole are illustrated in Figure 20.

If the collapse is not perfectly symmetric, any asymmetry in the resulting black hole is radiated away as gravitational waves, in such a way that the final result is a black hole that is completely characterized by the three parameters M, J, and Q. The black hole, once formed, has no hints about the details of the formation process and its previous history.

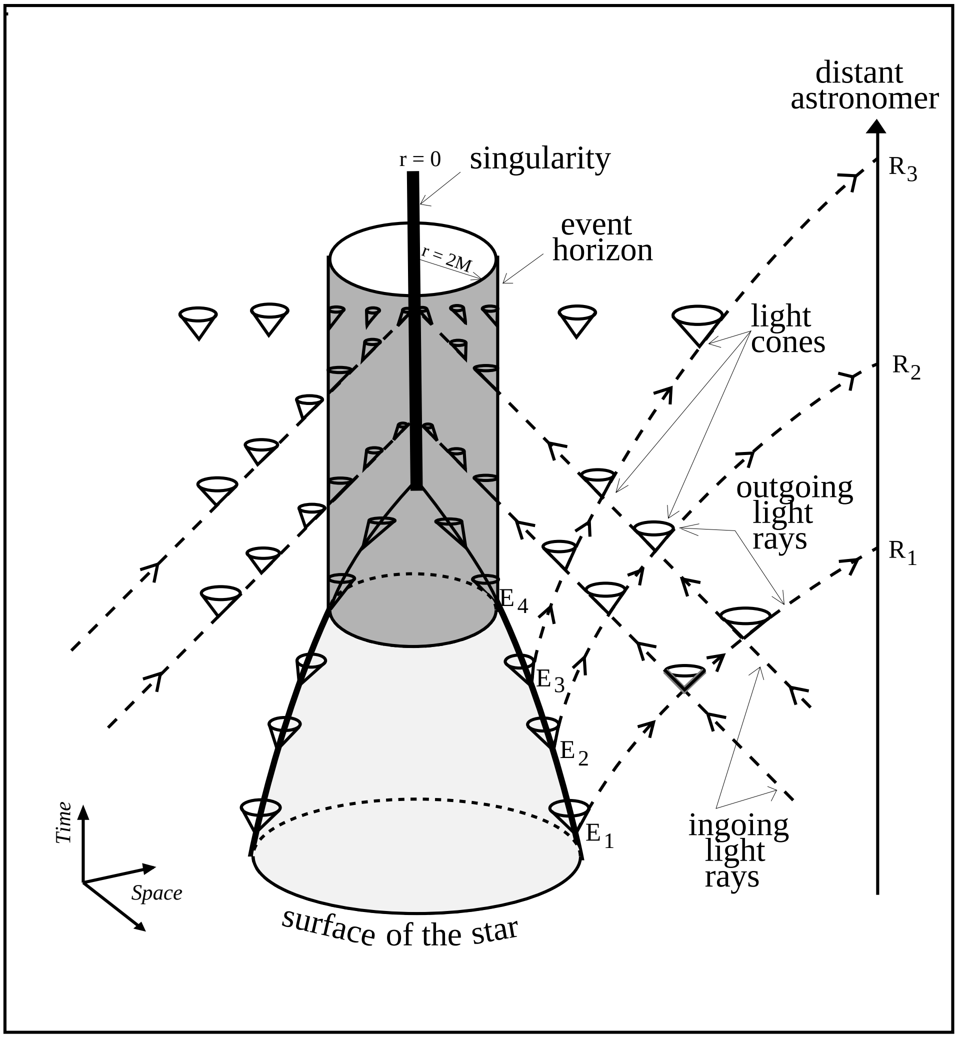

Figure 19. An Eddington-Finkelstein diagram of a collapsing star with the subsequent black hole formation. Adapted from J-P. Luminet (1998).

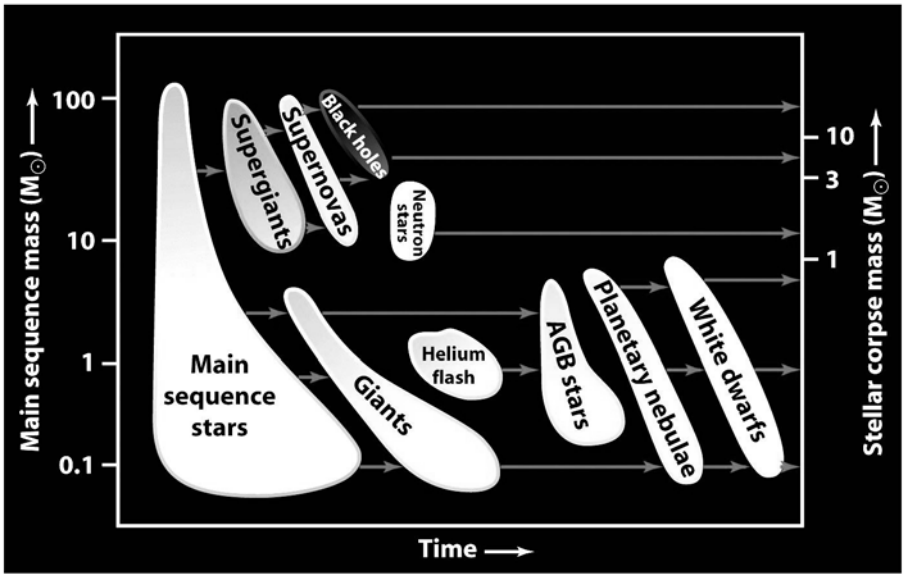

Figure 20. Life cycle of stars and channels for black hole formation.

Gravitational collapse can also be the result of inhomogeneities in the original metric giving rise to mini-black holes as proposed by Hawking (1971), although the number of microscopic black holes is strongly constrained by observations of cosmic gamma-ray background emission.

Finally, supermassive black holes can result from a variety of processes occurring at the center of galaxies as discussed by Rees (1984). See Figure 21 for a sketch of some possible formation paths. Some current views, however, suggest that galaxies were formed around seed massive black holes which were the result of the gravitational collapse of dark matter.

20. Black hole thermodynamics

The area of a Schwarzschild black hole is:

$$A_{\rm Schw} = 4\pi r_{\rm Schw}^2 = \frac{16\pi G^2 M^2}{c^4}. \tag{247}$$

In the case of a Kerr-Newman black hole, the area is:

$$\begin{aligned} A_{\rm KN} &= 4\pi\left(r_+^2 + \frac{a^2}{c^2}\right) \\ &= 4\pi\left[\left(\frac{GM}{c^2} + \frac{1}{c^2}\sqrt{G^2M^2 - GQ^2 - a^2}\right)^2 + \frac{a^2}{c^2}\right]. \end{aligned} \tag{248}$$

Notice that expression (248) reduces to (247) for $a = Q = 0$.

When a black hole absorbs a mass δM, its mass increases to $M + \delta M$, and hence, the area increases as well. Since the horizon can be crossed in just one

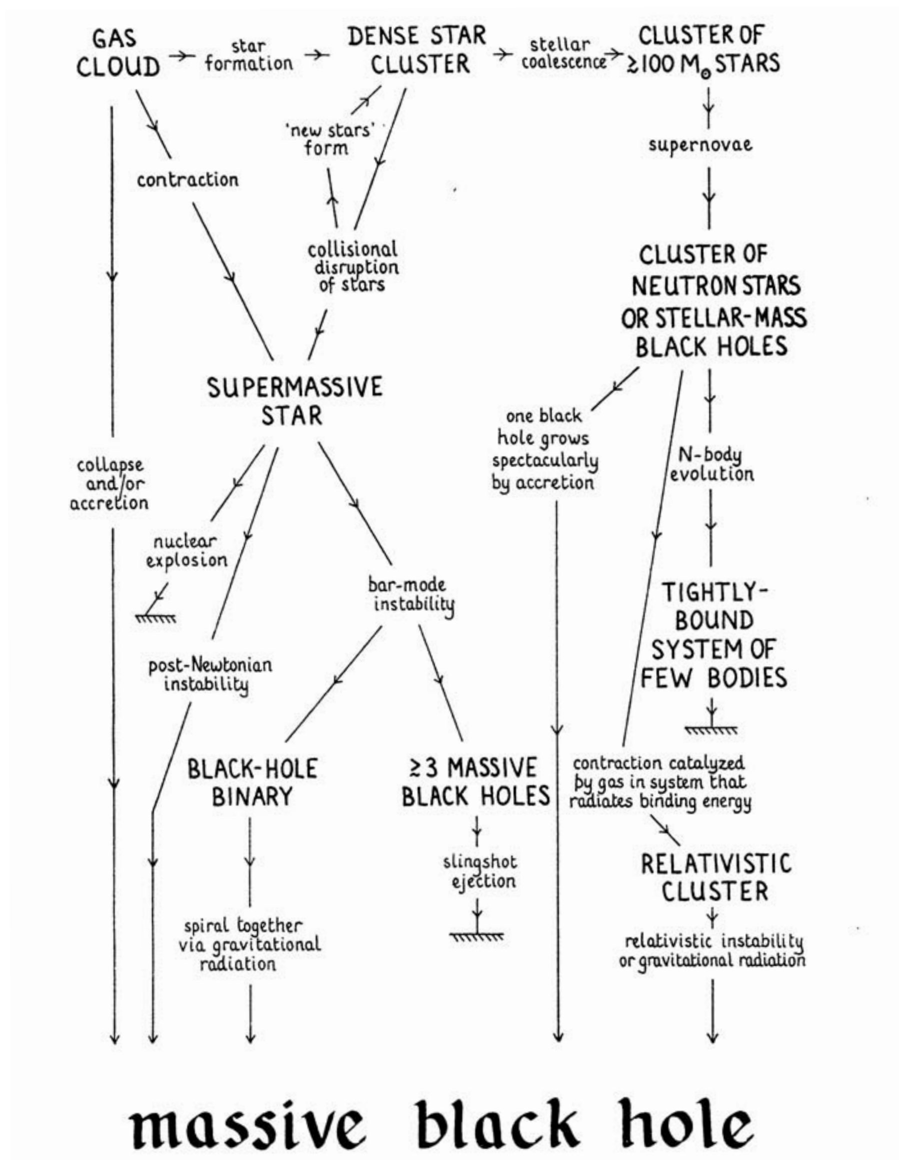

Figure 21. Massive and supermassive black hole formation channels. From Rees (1984).

direction the area of a black hole can only increase. This suggests an analogy with entropy. A variation in the entropy of the black hole will be related to the heat (δQ) absorbed through the following equation:

$$\delta S = \frac{\delta Q}{T_{\rm BH}} = \frac{\delta M c^2}{T_{\rm BH}}. \tag{249}$$

Particles trapped in the black hole will have a wavelength:

$$\lambda = \frac{\hbar c}{kT} \propto r_{\rm Schw}, \tag{250}$$

where k is the Boltzmann constant. Then,

$$\xi \frac{\hbar c}{kT} = \frac{2GM}{c^2},$$

where ξ is a numerical constant. Then,

$$T_{\rm BH} = \xi \frac{\hbar c^3}{2GkM}, \quad \text{and} \quad S = \frac{c^6}{32\pi G^2 M} \int \frac{dA_{\rm Schw}}{T_{\rm BH}} = \frac{c^3 k}{16\pi \hbar G \xi} A_{\rm Schw} + \text{ constant}.$$

A quantum mechanical calculation of the horizon temperature in the Schwarzschild case leads to $\xi = (4\pi)^{-1}$ and hence:

$$T_{\rm BH} = \frac{\hbar c^3}{8GMk} \cong 10^{-7}\text{K} \left(\frac{M_\odot}{M}\right). \tag{251}$$

We can write then the entropy of the black hole as:

$$S = \frac{kc^3}{4\pi \hbar G} A_{\rm Schw} + \text{ constant} \sim 10^{77} \left(\frac{M}{M_\odot}\right)^2 k \text{ JK}^{-1}. \tag{252}$$

The formation of a black hole implies a huge increase of entropy. Just to compare, a solar mass star has an entropy $\sim$ 20 orders of magnitude lower. This tremendous increase of entropy is related to the loss of all the structure of the original system (e.g. a star) once the black hole is formed.

The analogy between area and entropy allows to state a set of laws for black holes thermodynamics:

- First law (energy conservation): $dM = T_{\rm BH} dS + \Omega_+ dJ + \Phi dQ$. Here, Ω is the angular velocity and Φ the electrostatic potential.

- Second law (entropy never decreases): In all physical processes involving black holes the total surface area of all the participating black holes can never decrease.

- Third law (Nernst's law): The temperature (surface gravity) of a black black hole cannot be zero. Since $T_{\rm BH} = 0$ with $A \neq 0$ for extremal charged and extremal Kerr black holes, these are thought to be limit cases that cannot be reached in Nature.

- Zeroth law (thermal equilibrium): The surface gravity (temperature) is constant over the event horizon of a stationary axially symmetric black hole.

21. Quantum effects in black holes

If a temperature can be associated with black holes, then they should radiate as any other body. The luminosity of a Schwarzschild black hole is:

$$L_{\rm BH} = 4\pi r_{\rm Schw}^2 \sigma T_{\rm BH}^4 \sim \frac{16\pi\sigma\hbar^4 c^6}{(8\pi)^4 G^2 M^2 k^4}. \tag{253}$$

Here, σ is the Stephan-Boltzmann constant. This expression can be written as:

$$L_{\rm BH} = 10^{-17} \left(\frac{M_\odot}{M}\right)^2 \quad \text{erg s}^{-1}. \tag{254}$$

The lifetime of a black hole is:

$$\tau \cong \frac{M}{dM/dt} \sim 2.5 \times 10^{63} \left(\frac{M}{M_\odot}\right)^3 \quad \text{years}. \tag{255}$$

Notice that the black hole heats up as it radiates!. This occurs because when the hole radiates, its mass decreases and then according to Eq. (251) the temperature must rise.

If nothing can escape from black holes because of the existence of the event horizon, what is the origin of this radiation?. The answer, found by Hawking (1974), is related to quantum effects close to the horizon. According to the Heisenberg relation $\Delta t \Delta E \geq \hbar/2$ particles can be created out of the ground state of a quantum field as far as the relation is not violated. Particles must be created in pairs, along a tiny time, in order to satisfy conservation laws other than energy. If a pair is created close to the horizon an one particle crosses it, then the other particle can escape provided its momentum is in the outward direction. The virtual particle is then transformed into a real particle, at expense of the black hole energy. The black hole then will lose energy and its area will decrease slowly, violating the second law of thermodynamics. However, there is no violation if we consider a *generalized second law*, that always holds: *In any process, the total generalized entropy $S + S_{\rm BH}$ never decreases* (Bekenstein 1973).

22. Black hole magnetospheres

In the real universe black holes are not expected to be isolated, hence the ergosphere should be populated by charged particles. This plasma would rotate in the same sense as the black hole due to the effects of the frame dragging. A magnetic field will develop and will rotate too, generating a potential drop that might accelerate particles up to relativistic speed and produce a wind along the rotation axis of the hole. Such a picture has been consistently developed by Punsly and Coroniti (1990a, b) and Punsly (2001).

In Figures 22 and 23 we show the behavior of fields and currents in the ergosphere. Since the whole region is rotating, an ergospheric wind arises along the direction of the large scale field.

Blandford and Znajek (1977) developed a general theory of force-free steady-state axisymmetric magnetosphere of a rotating black hole. In an accreting black

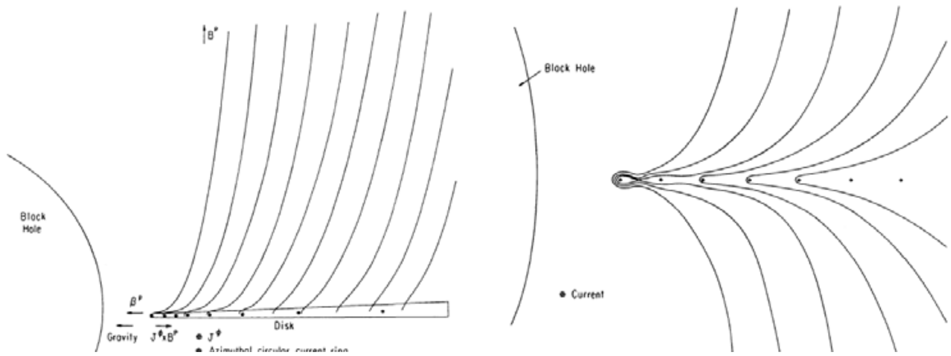

Figure 22. Currents in the infalling matter supports a radial magnetic field. As the inner part of the current sheet approaches to the black hole the sources are redshifted to observers at infinity and their contribution to the poloidal magnetic field diminish. At some point X, the field reconnects. From Punsly & Coroniti(1990a).

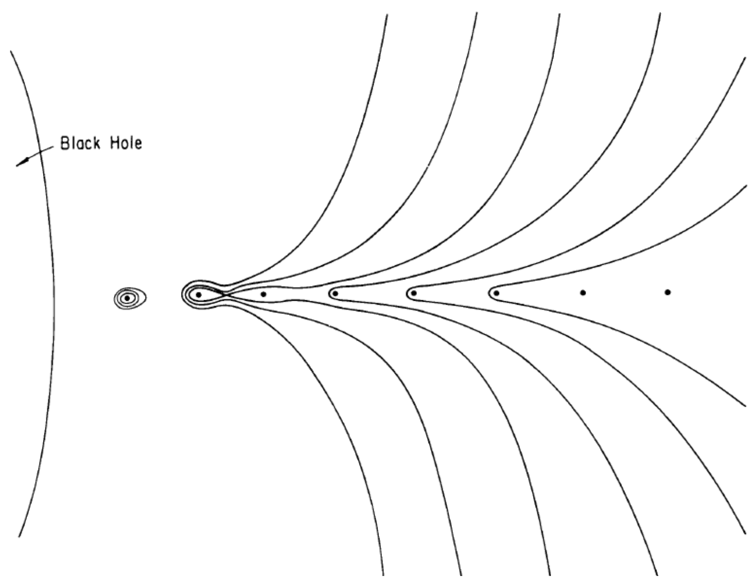

Figure 23. As reconnection proceeds, the magnetic field around the innermost currents is disconnected from the large scale field allowing the destruction of the magnetic flux by the black hole. From Punsly & Coroniti(1990a).

hole, a magnetic field can be sustained by external currents, but as such currents move along the horizon, the field lines are usually representing as originating from the horizon and then being torqued by rotation. The result is an outgoing electromagnetic flux of energy and momentum. This picture stimulated the development of the so-called "membrane paradigm" by Thorne et al. (1986) where the event horizon is attributed with a set of physical properties. This model of black hole has been subjected to strong criticism by Punsly (2001) since General Relativity implies that the horizon is causally disconnected from the outgoing wind.

Recent numerical simulations (e.g. Komissarov 2004) show that the key role in the electrodynamic mechanisms of rotating black holes is played by the ergosphere and not by the horizon. However, globally the Blandford-Znajek solution seems to be asymptotically stable. The twisted magnetic fields in the ergosphere of a Kerr black hole are shown in Figures 24, 25 and 26. The controversy still goes on and a whole bunch of new simulations are exploring the different aspects of relativistic magneto-hydrodynamic (MHD) outflows from black hole magnetospheres. We will say more about these outflows when discussing astrophysical jets.

23. Back hole interiors

The most relevant feature of black hole interiors is that the roles of space and time are exchanged: the space radial direction becomes time, and time becomes a space direction. Inside a spherical black hole, the radial coordinate becomes *time-like*: changes occur in a prefer direction, i.e. toward the space-time singularity. This means that the black hole interior is essentially dynamic. In order to see this, let us recall the Schwardzschild metric (139):

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

If we consider a radially infalling test particle:

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2.$$

The structure of the light cones is defined by the condition $ds = 0$. Writing $r_{\rm Schw}$ once again for the Schwarzschild radius, we get:

$$\left(1 - \frac{r_{\rm Schw}}{r}\right) c^2 dt^2 - \left(1 - \frac{r_{\rm Schw}}{r}\right)^{-1} dr^2 = 0. \tag{256}$$

If we consider the interior of the black hole, $r < r_{\rm Schw}$. Then,

$$\left(1 - \frac{r_{\rm Schw}}{r}\right)^{-1} dr^2 - \left(1 - \frac{r_{\rm Schw}}{r}\right) c^2 dt^2 = 0. \tag{257}$$

The signs of space and time are now exchanged. The light cones, that in Schwarzschild coordinates are shown in Fig. 5, are now oriented with the time

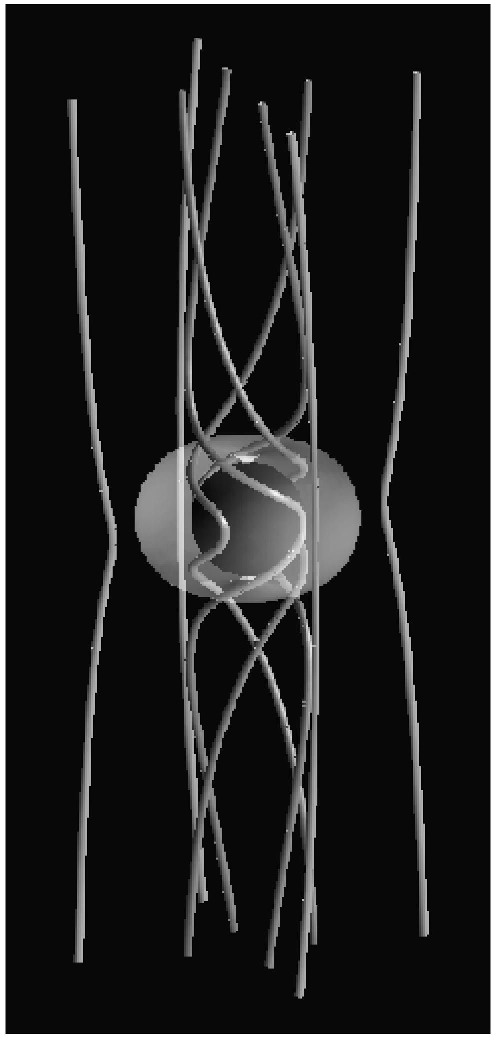

Figure 24. Effects of the ergosphere of a Kerr black hole on external magnetic field lines. Credit NASA Jet Propulsion Laboratory (NASA-JPL).

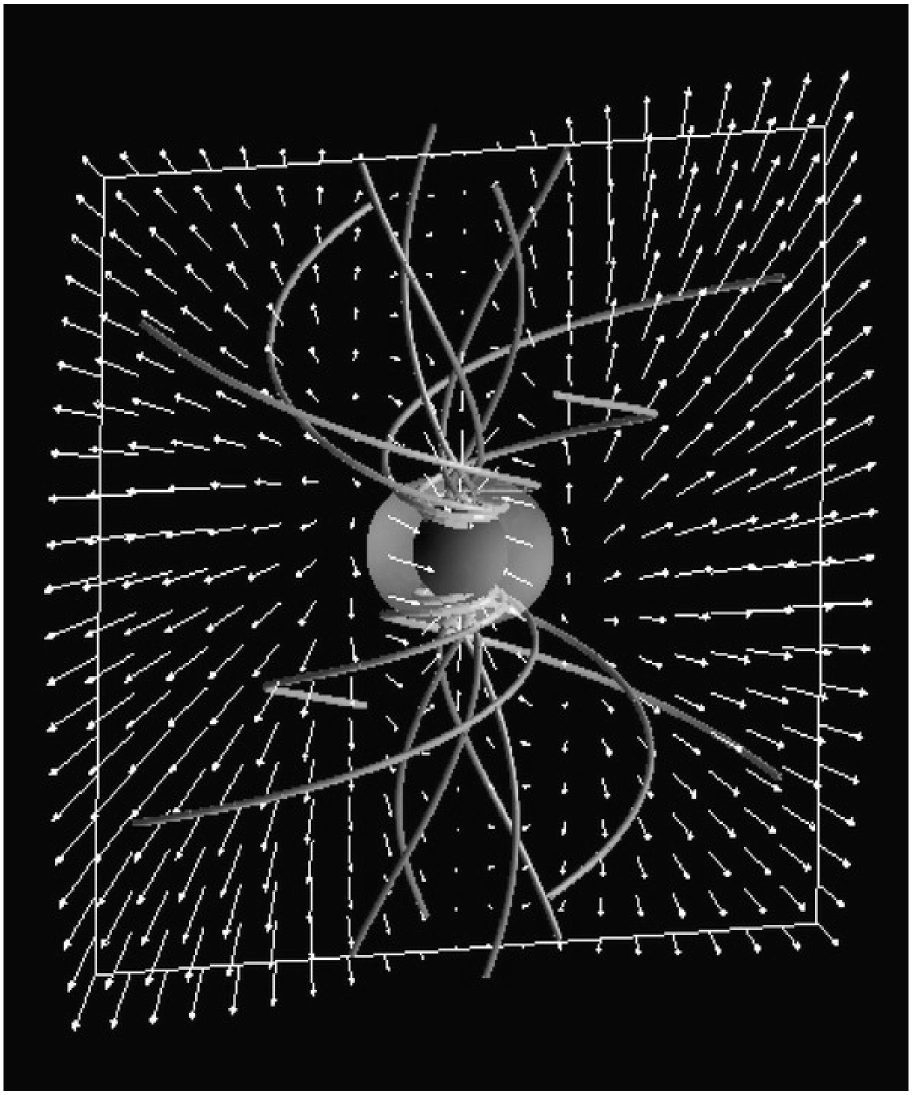

Figure 25. Three-dimensional graphic of magnetic field lines and plasma flow around the Kerr black hole. The black sphere at the center depicts the black hole horizon. The transparent (gray) surface around the black hole is that of the ergosphere. The arrows show the plasma flow velocity. The tubes in the shape of propellers show the magnetic field lines. From Koide (2004).

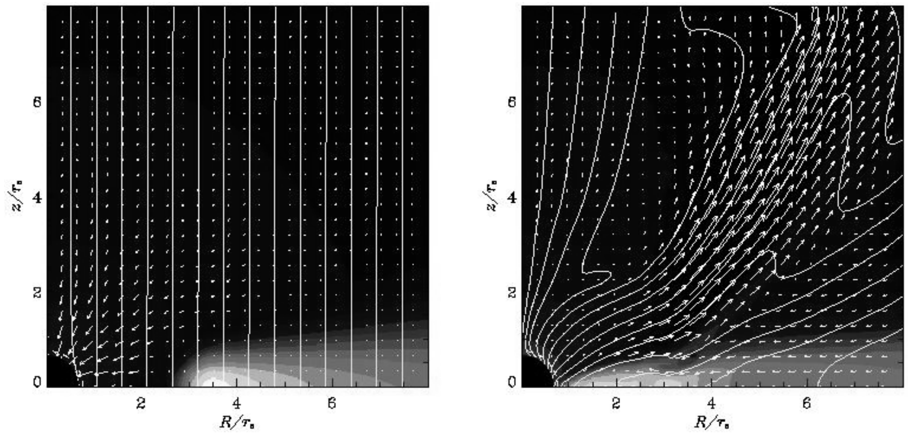

Figure 26. Black hole magnetosphere (corona) and resulting outflows. From Koide et al. (1999).

axis perpendicular to the event horizon. The trajectory of photons is given by:

$$\frac{dr}{dt} = \mp c \left| 1 - \frac{r_{\rm Schw}}{r} \right|, \tag{258}$$

with r always decreasing. The light cones are thinner and thinner as r gets closer to the singularity at $r = 0$. In addition to infalling particles, there is a small flux of gravitational radiation into the black hole through the horizon because of small perturbations outside it. This radiation, as the material particles and photons, ends in the singularity.

In the case of a Kerr black hole, between the two horizons space and time also exchange roles as it happens with the Schwarzschild interior black hole space-time. Instead of time always moving inexorably onward, the radial dimension of space moves inexorably inward to the second horizon, that it is also a Cauchy horizon, i.e. a null hyper-surface beyond which predictability breaks down. After that, the Kerr solution predicts a second reversal so that one can avoid the ring singularity and achieve to orbit safely. In this strange region inside the Cauchy horizon the observer can, by selecting a particular orbit around the ring singularity, travel backwards in time and meet himself, i.e. there are closed time-like curves. Another possibility admitted by the equations for the observer in the central region is to plunge through the hole in the ring to emerge in an anti-gravity universe, whose physical laws would be even most peculiar. Or he can travel through two further horizons, (or more properly anti-horizons), to emerge at coordinate time $t = \infty$ into some other universe. All this can be represented in a Penrose-Carter diagram for a Kerr black hole (see Figure 27).

All the above discussion on Kerr black hole interiors is rather academic, since in real black holes the inner horizon is likely unstable. Poisson and Israel (1990) have shown that when the space-time is perturbed by a fully non-linear, ingoing, spherically-symmetric null shell, a null curvature singularity develops at the inner horizon. This singularity is "weak" in the sense that none the scalar curvature invariants is divergent there. The singularity development at the Cauchy horizon

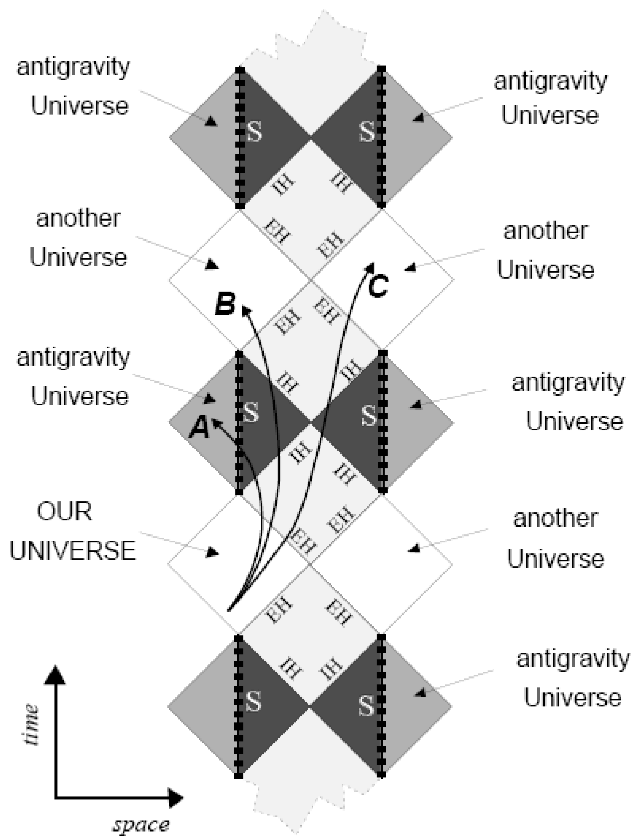

Figure 27. Penrose-Carter diagram of a non-extreme Kerr solution. The figure is repeated infinitely in both direction. One trajectory ends in the singularity (A), the other two (B and C) escape. IH stands for “inner horizon”, EH for “external horizon”, and S for “singularity”. Adapted from J.-P. Luminet (1997).

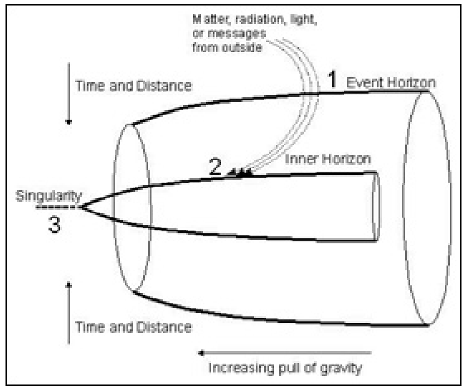

Figure 28. Diagram representing a Kerr black hole interior and the accumulation of energy-momentum at the inner horizon.

would deal off the "Kerr tunnel" that would lead to other asymptotically flat universes. The key factor producing the instability is the infinite concentration of energy density close to the Cauchy horizon as seen by a free falling observer. The infinite energy density is due to the ingoing gravitational radiation, which is partially backscattered by the inner space-time curvature. The non-linear interaction of the infalling and outgoing gravitational fluxes results in the weak curvature singularity on the Cauchy horizon, where a tremendous inflation of the mass parameter takes place (see Figure 28). This changes the conception of the Kerr black hole interior, since instead of a Cauchy horizon acting as a curtain beyond which predictability breaks down we have a microscopically thin region near the inner horizon where the curvature is extremely high (Poisson and Israel 1990). Other analysis based on plane-symmetric space-time analysis seem to suggest that instead of a null, weak singularity a space-like strong singularity is formed under generic non-linear perturbations (Yurtsever 1993). This is the same result that can be obtained through a linear perturbation analysis of the inner horizon. More recent numerical investigations using regular initial data find a mass inflation-type null singularity (Droz 1997).

The issue of realistic black hole interiors is still an open one, with active research ongoing.

24. Singularities

A space-time is said to be *singular* if the manifold M that represents it is *incomplete*. A manifold is incomplete if it contains at least one *inextendible* curve. A curve $\gamma : [0, a) \longrightarrow M$ is inextendible if there is no point p in M such that $\gamma(s) \longrightarrow p$ as $a \longrightarrow s$, i.e. γ has no endpoint in M. A given space-time (M, g_{ab}) has an *extension* if there is an isometric embedding $\theta : M \longrightarrow M'$, where (M', g'_{ab}) is a space-time and θ is onto a proper subset of M'. A space-time is *singular* if it contains a curve γ that is inextendible in the sense given above. Singular space-times contain singularities.

A so-called *coordinate singularity* is not as real singularity. It seems to be singular in some space-time representation but it can be removed by a coordinate change, like the "Schwarzschild singularity" at $r_{\rm Scwh} = 2GM/c^2$ in a Schwarzchild space-time. We can change to Eddington-Finkelstein coordinates, for instance, and then we see that geodesic lines can go through that point of the manifold. Essential singularities cannot be removed in this way. This occurs, for instance, with the singularity at $r = 0$ in the Schwarzschild space-time or with the ring singularity at $r = 0$ and $\theta = \pi/2$ in the Kerr metric written in Boyer-Lindquist coordinates[11]. In such cases, the curvature scalar $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ diverges. There is no metric there, and the Einstein equations cannot be defined.

An essential or true singularity should not be interpreted as a representation of a physical object of infinite density, infinite pressure, etc. Since the singularity does not belong to the manifold that represents space-time in General Relativity, it simply cannot be described or represented in the framework of such a theory. General Relativity is incomplete in the sense that it cannot provide a full description of the gravitational behavior of any physical system. True singularities are not within the range of values of the bound variables of the theory: they do not belong to the ontology of a world that can be described with 4-dimensional differential manifolds.

An essential singularity in solutions of the Einstein field equations is one of two things:

1. A situation where matter is forced to be compressed to a point (a space-like singularity).

2. A situation where certain light rays come from a region with infinite curvature (time-like singularity).

Space-like singularities are a feature of non-rotating uncharged black-holes, whereas time-like singularities are those that occur in charged or rotating black hole exact solutions, where time-like or null curves can always avoid hitting the singularities.

Singularities do not belong to classical space-time. This is not surprising since singularities are extremely compact systems. At such small scales, relations among things should be described in a quantum mechanical way. If space-time is formed by the events that occur to things, it should be represented through quantum mechanic theory when the things are described by a quantum theory. Since even in the standard quantum theory time appears as a continuum variable, a new approach is necessary.

Space-time singularities are expected to be covered by horizons. Although formation mechanisms for naked singularities have been proposed, the following conjecture is usually considered valid:

- Cosmic Censorship Conjecture (Roger Penrose): Singularities are always hidden behind event horizons.

We emphasize that this conjecture is not proved in General Relativity and hence it has not the strength of a theorem of the theory.

---

[11]In Cartesian coordinates the Kerr singularity occurs at $x^2 + y^2 = a^2c^{-2}$ and $z = 0$.

The classical references on singularities are Hawking and Ellis (1973) and Clarke (1993).

Singularity theorems

Several singularity theorems can be proved from pure geometrical properties of the space-time model (Clarke 1993). The most important one is due to Hawking & Penrose (1970):

Theorem. Let $(M,\ g_{ab})$ a time-oriented space-time satisfying the following conditions:

1. $R_{ab}V^aV^b \geq 0$ for any non space-like V^a.
2. Time-like and null generic conditions are fulfilled.
3. There are no closed time-like curves.
4. At least one of the following conditions holds
   - There exists a compact[12] achronal set[13] without edge.
   - There exists a trapped surface.
   - There is a $p \in M$ such that the expansion of the future (or past) directed null geodesics through p becomes negative along each of the geodesics.

Then, $(M,\ g_{ab})$ contains at least one incomplete time-like or null geodesic.

The theorem is purely geometric, no physical law is invoked. If the theorem has to be applied to the physical world, the hypothesis must be supported by empirical evidence. Condition 1 will be satisfied if if the energy-momentum T^{ab} satisfies the so-called *strong energy condition*: $T_{ab}\xi^a\xi^b \geq -(1/2)T^a_a$, for any time-like vector ξ^a. If the energy-momentum is diagonal: $T_{\mu\mu} = (\rho, -p, -p, -p)$ the strong enegy condition can be written as: $\rho + 3p \geq 0$ and $\rho + p \geq 0$. Condition 2 requires that any time-like or null geodesic experiences a tidal force at some point in its history. Condition 4a requires that, at least at one time,

[12] A space is said to be compact if whenever one takes an infinite number of "steps" in the space, eventually one must get arbitrarily close to some other point of the space. Thus, whereas disks and spheres are compact, infinite lines and planes are not, nor is a disk or a sphere with a missing point. In the case of an infinite line or plane, one can set off making equal steps in any direction without approaching any point, so that neither space is compact. In the case of a disk or sphere with a missing point, one can move toward the missing point without approaching any point within the space. More formally, a topological space is compact if, whenever a collection of open sets covers the space, some subcollection consisting only of finitely many open sets also covers the space. A topological space is called compact if each of its open covers has a finite subcover. Otherwise it is called non-compact. Compactness, when defined in this manner, often allows one to take information that is known locally – in a neighborhood of each point of the space – and to extend it to information that holds globally throughout the space.

[13] A set of points in a space-time with no two points of the set having time-like separation.

the universe is closed and the compact slice that corresponds to such a time is not intersected more than once by a future directed time-like curve. The trapped surfaces mentioned in 4b refers to horizons due to gravitational collapse. Condition 4c requires that the universe is collapsing in the past or the future.

A closely related theorem is due to Hawking (1967):

Theorem. Let $(M,\ g_{ab})$ a time-oriented space-time satisfying the following conditions:

1. $R_{ab}V^aV^b \geq 0$ for any non space-like V^a.
2. There exists a compact space-like hypersurface $\Sigma \subset M$ without edge.
3. The unit normals to Σ are everywhere converging (or diverging).

Then, $(M,\ g_{ab})$ is time-like geodesically incomplete.

Although singularity theorems apply to spherically symmetric black holes, they do not seem to apply to the Universe as a whole.

25. Accretion onto black holes

Accretion is the process of matter falling into the potential well of a gravitating object. The accretion of matter which does not have angular momentum is basically determined by the relation between the sound speed $a_{\rm s}$ in the matter and the velocity of the object respect to the medium $v_{\rm rel}$ (Bondi & Hoyle 1944). The accretion of matter with angular momentum can lead to the formation of an accretion disk around the compact object (Shakura & Sunyaev 1973). In what follows, we will discuss the fluid dynamics of accretion onto a black hole.

The are four basic regimes of accretion onto a black hole:

- Spherical symmetric accretion. It occurs when $v_{\rm rel} << a_{\rm s}$ and the accreting matter does not have any significant angular momentum.
- Cylindrical accretion. The angular momentum of the medium remains small but now $v_{\rm rel} \geq a_{\rm s}$.
- Disk accretion. The total angular momentum of matter is enough as to form an accretion disk around the black hole.
- Two-stream accretion. Both a quasi-spherically symmetric inflow of matter coexists with disk accretion (e.g. Narayan & Yi 1994).

The fluid dynamic description of a physical process is applicable if the free mean path for particles in the medium is much shorter than the typical size-scale of the system. In the case of accretion, the self-gravitation of the accreting matter is usually negligible, so the characteristic length is the gravitational capture radius $R_{\rm G}$. This quantity is equal to the distance at which the kinetic energy of the matter is of the order of the gravitational energy:

$$\frac{1}{2}(a_{\rm s}^2 + v_{\rm rel}^2) = \frac{GM}{R_{\rm G}}.$$

Here, M is the mass of the accreting object. Hence,

$$R_{\rm G} = \frac{2GM}{a_{\rm s}^2 + v_{\rm rel}^2}. \tag{259}$$

In the absence of radiation, the equations that fully describe the accretion process are:

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}\,\vec{\nabla}\vec{v} = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\phi, \tag{260}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho\vec{v}) = 0, \tag{261}$$

$$P = P(\rho), \tag{262}$$

$$\nabla^2\phi = -4\pi G\rho_{\rm tot}. \tag{263}$$

Equation (260) is the non-relativistic Euler's equation in a gravitational field of potential ϕ, with P being the fluid pressure and ρ its density. Equation (315) is the continuity equation. The third equation is the equation of state of the fluid. The final equation is Poisson's equation for the gravitational potential produced by the total mass density of the system. By the moment we assume non-radiative and non-relativistic accretion.

The accretion rate onto the black hole will be:

$$\dot{M} = \sigma_{\rm G}\rho v_{\rm rel}, \tag{264}$$

where $\sigma_{\rm G}$ is the cross section of gravitational capture. This cross section strongly depends on the nature of the gas. If the gas is formed by collisionless particles, these can be captured only if they have an impact parameter smaller than the size of the compact object, whereas if the gas is a continuous media, angular momentum is not conserved and essentially all particles with kinetic energy smaller than the gravitational energy will be captured. Considering a Schwarzschild black hole, then, we have:

$$\frac{\sigma_{\rm G(collisionless)}}{\sigma_{\rm G(continuous)}} = \frac{R_{\rm Schw}}{R_{\rm G}} << 1. \tag{265}$$

Under typical conditions in the interstellar medium the accretion of fluid is about a million times the accretion of collisionless particles. The accretion rate of Eq. (264) can be written is the case of fluid dynamics as:

$$\dot{M} = \pi R_{\rm G}^2 \rho v_{\rm rel}. \tag{266}$$

25.1. Spherically symmetric accretion

Bondi (1952) was the first to obtain the stationary spherically symmetric solution of Eqs (260-263), assuming an adiabatic equation of state $P \propto \rho^{\gamma}$. The Euler equation implies the conservation of energy:

$$\frac{v^2}{2} + \frac{\gamma}{\gamma - 1}\frac{P}{\rho} - \frac{GM}{R} = \text{ constant} = \epsilon_0. \tag{267}$$

The continuity equation can written as:

$$\dot{M} = 4\pi R^2 \rho v = \text{ constant}, \tag{268}$$

where we have considered accretion over a sphere of radius R.

The boundary conditions at infinity imply:

$$\epsilon_0 = \frac{\gamma}{\gamma - 1}\frac{P_\infty}{\rho_\infty} = \frac{a_\infty^2}{\gamma - 1}. \tag{269}$$

The speed of sound is

$$a_{\rm s} = \sqrt{\frac{\partial P}{\partial \rho}} = \sqrt{\frac{\gamma P}{\rho}}.$$

Then,

$$\frac{v^2}{2} + \frac{a_{\rm s}^2}{\gamma - 1} = \frac{GM}{R} + \frac{a_\infty^2}{\gamma - 1}, \tag{270}$$

and

$$v = \frac{\dot{M}}{4\pi\rho_\infty R}\left(\frac{a_\infty}{a_{\rm s}}\right)^{2/(\gamma-1)}. \tag{271}$$

There is a critical point at $R = R_{\rm s}$ which the gas velocity overcomes the speed of the sound. At $R << R_{\rm s}$ the matter is practically in a state of free fall toward the black hole. This is because of the flow becomes supersonic and then the underlying layers do not do not affect the entrained matter. Then, we can write:

$$v \approx \sqrt{\frac{2GM}{R}}, \tag{272}$$

and

$$\rho \approx \frac{\dot{M}}{4\pi\sqrt{2GM}} R^{-3/2}. \tag{273}$$

If the effect of radiation is going to be taken into account, then we add the second law of thermodynamics to Eqs. (260-263). The energy variation per unit mass is:

$$d\epsilon = -PdV + dQ, \tag{274}$$

where V is the specific gas volume and dQ is the heat released by unit mass. For the case of a monoatomic gas or a fully ionized plasma, we have:

$$\frac{3}{2\mu} R_{\rm u}\frac{dT}{dt} = R_{\rm u}\frac{T}{\mu\rho}\frac{d\rho}{dt} - \alpha_{\rm ff} T^{1/2}\rho + \frac{dQ}{dt}. \tag{275}$$

Here, $R_{\rm u}$ is the universal gas constant, the second term on the right is due to Bremsstrahlung losses ($\alpha_{\rm ff} \approx 5 \times 10^{20}$ erg g^{-1} s^{-1}), and the third term takes into account other possible radiation losses. Using $vdt = dR$ and reminding that

$\rho \propto R^{-3/2}$, we can obtain the equation for the temperature distribution in a steady state spherically symmetric accretion flow:

$$\frac{dT}{dR} = -\frac{T}{R} + \text{ constant } \frac{\sqrt{T}}{R} + \frac{2\mu}{3R_{\rm u}}\frac{dQ}{dR}. \tag{276}$$

Notice that if there are no additional radiation losses besides free-free radiation ($dQ/dR = 0$), Eq. (276) can be solved to obtain

$$T = \left[K \ln\left(\frac{R}{R_{\rm G}}\right) + T_{\infty}^{1/2}\right]^2, \tag{277}$$

where we have assumed that at $R = R_{\rm G}$ the matter temperature is T_{∞}. Eq. (277) shows that under such conditions the temperature *decreases* as the flow approaches to the black hole. A fluid that behaves in this way is called a *cooling flow*.

The radial free fall time is:

$$t_{\rm fall} \approx \frac{R}{v_R} \propto R^{3/2}, \tag{278}$$

whereas the cooling time for free-free losses ($dQ/dT \propto T^{1/2}\rho$) is[14]:

$$t_{\rm cool} = \frac{3R_{\rm u}T}{2dQ/dt} \propto \frac{\sqrt{T}}{\rho} \approx R. \tag{279}$$

Comparing both equations we see that the relative role of cooling decreases as the black hole is approached.

Close to the black hole, however, there could be sources of radiation, if magnetic field and angular momentum are involved. The outgoing radiation will pass through the accretion flow and can influence its dynamics. Let σ be the cross section of interaction of the emitted radiation with matter.The force acting upon the infalling particles is:

$$F_{\rm rad} = \frac{\sigma L}{4\pi R^2 c}, \tag{280}$$

where L is the luminosity of the radiation. The attractive gravitational force on a particle of mass m_p is:

$$F_{\rm grav} = \frac{GMm_p}{R^2}, \tag{281}$$

with M the mass of the black hole. For a luminosity $L = L_{\rm Edd}$ these forces are balanced and spherical accretion is stopped:

$$L_{\rm Edd} = \frac{4\pi M m_p c}{\sigma}. \tag{282}$$

[14]Notice that $T \propto R^{-1}$ and $\rho \propto R^{3/2}$.

This critical luminosity is called the Eddington luminosity of the accreting source. In the case of Thomson scattering $\sigma = \sigma_{\rm T} \approx 0.66 \times 10^{-24}$ cm^2, and we can write:

$$L_{\rm Edd} \approx 1.3 \times 10^{38} \left(\frac{M}{M_\odot}\right) \;\; \text{erg s}^{-1}. \tag{283}$$

Associated with the Eddington luminosity we have a critical accretion rate:

$$\dot{M}_{\rm Edd} = \frac{L_{\rm Edd}}{c^2} \approx 0.2 \times 10^{-8} \left(\frac{M}{M_\odot}\right) \;\; M_\odot \text{ yr}^{-1}. \tag{284}$$

The Eddington temperature $T_{\rm Edd}$ is the characteristic black body temperature required for a body with the Schwarzschild radius to radiate $L_{\rm Edd}$:

$$T_{\rm Edd} = \left(\frac{L_{\rm Edd}}{4\pi\sigma_{\rm SB} R^2_{\rm Schw}}\right) \approx 6.6 \times 10^7 \left(\frac{M}{M_\odot}\right)^{-1/4} \;\; \text{K}. \tag{285}$$

Another way to inhibit spherical accretion is the production of winds or particle ejection in the inner accreting regions. If $L_{\rm ej}$ is the power carried away by the ejected particles and $v_{\rm ej}$ is their velocity, the exerted pressure will be:

$$P_{\rm ej} = \frac{L_{\rm ej}}{4\pi R^2 v_{\rm ej}}. \tag{286}$$

If the central source ejects particles before the onset of the spherical accretion, pressures must be equated at the gravitational capture radius to find the critical luminosity in ejected particles:

$$\frac{L_{\rm ej}}{4\pi R^2_{\rm G} v_{\rm ej}} = \rho v^2_\infty = \frac{\dot{M}}{4\pi R^2_{\rm G}} v_\infty.$$

From here we get:

$$L^{\rm crit}_{\rm ej} = \dot{M} v_\infty v_{\rm ej}. \tag{287}$$

Using the fact that a fraction η of the accretion power is released as radiation, i.e.

$$L = \eta \dot{M} c^2, \tag{288}$$

we obtain

$$L^{\rm crit}_{\rm ej} = \frac{L}{\eta} \left(\frac{v_\infty v_{\rm ej}}{c^2}\right). \tag{289}$$

It can be seen that a weak wind can stop the the spherical accretion.

25.2. Cylindrical accretion

The problem of cylindrical accretion is the problem of the determination of the gas accretion onto a moving gravitating center. Unlike the case of spherical accretion, the problem is quite complex and there are not analytical solutions of it. In cylindrical accretion, the velocity of the compact object respect to the medium $v_{\rm rel}$ is not negligible. In order to deal with the problem is convenient to adopt a coordinate system centered in the moving object. The symmetry axis of

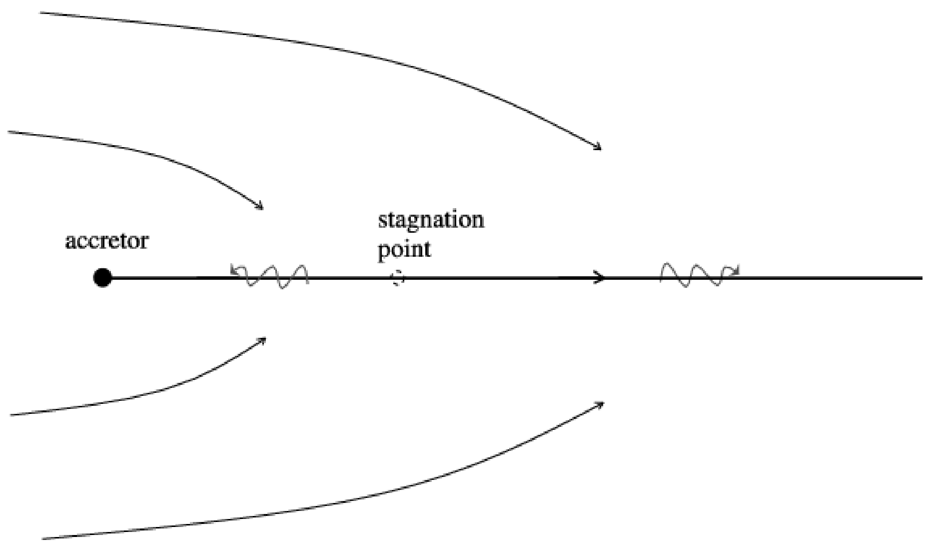

Figure 29. Stream lines around a moving compact object. Adapted from Foglizzo et al (2005).

the problem is determined by the line of motion of the object. The stream lines of the fluid then are hyperbolas centered in this axis. As individual particles move, the angular momentum is conserved relative to the accreting object (we assume no viscosity):

$$|R \times v| = bv_{\rm rel}, \tag{290}$$

where b is the impact parameter. The contribution to the accretion is due only to the component $v_\perp$, which is perpendicular to the symmetry axis. A picture of the situation is shown in Fig. 29, where the stagnation point is the point at which $v_{||} = 0$.

If $R_{\rm col}$ is the distance from the compact object on the symmetry axis:

$$v_\perp R_{\rm col} = bv_{\rm rel}. \tag{291}$$

The particles to be captured by the compact object are those for which the velocity is lower than the parabolic velocity:

$$v_{||} \leq \sqrt{\left(\frac{2GM}{R_{\rm col}}\right)}. \tag{292}$$

The conservation of energy implies:

$$\frac{1}{2}(v_{||}^2 + v_\perp^2) - \frac{GM}{R_{\rm col}} = \frac{1}{2}v_{\rm rel}^2. \tag{293}$$

It follows from Eqs. (292) and (293) that only particles satisfying $v_\perp \leq v_{\rm rel}$ will be captured. The gravitational capture cross section is determined by the capture radius:

$$\sigma_{\rm G} \approx \pi R_{\rm G}^2. \tag{294}$$

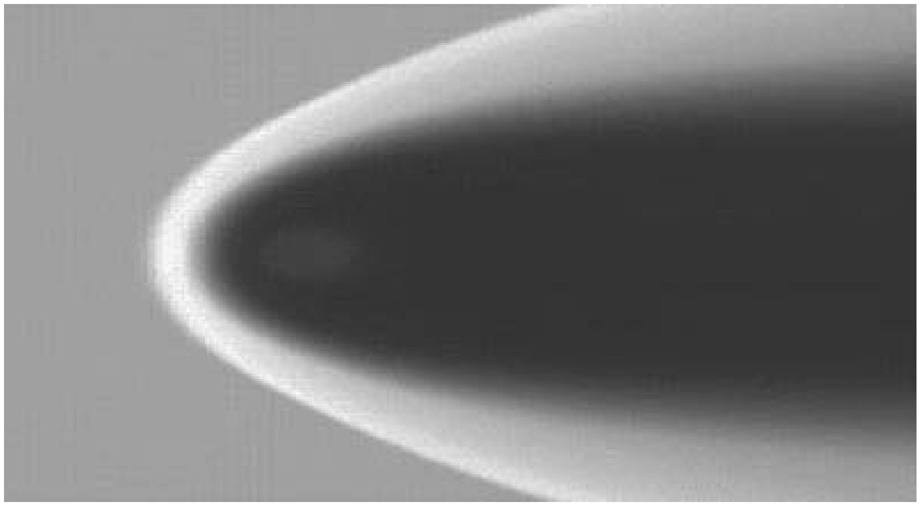

Figure 30. Simulation of a compact object moving supersonically in a fluid. A bow shock is formed around the moving object.

If we work in the fluid approximation, the supersonic motion of the compact object will lead to the formation of a bow shock (see Fig. 30).

A moving body through a gaseous medium with produce density perturbations. If these perturbations are small, we can write:

$$\rho = \rho_\infty + \delta\rho, \quad \delta = \frac{\delta\rho}{\rho_\infty}. \tag{295}$$

In addition, we assume that the gravitational center is point-like and moves at v_∞. The accretion rate is:

$$\dot{M} = \xi_1 \pi R_{\rm G}^2 v_\infty \rho_\infty,$$

wheres ξ_1 is a dimensionless parameter of the order of unity. Introducing the perturbation (295) in Eqs. (260-263) and linearizing, we get:

$$\frac{\partial v}{\partial t} = -a_\infty^2 \vec{\nabla}\delta + \vec{\nabla}\phi, \tag{296}$$

$$\frac{\partial \delta}{\partial t} + \vec{\nabla} v = -\xi_1 \pi R_{\rm G}^2 v_\infty \delta(R - v_\infty t), \tag{297}$$

$$\nabla^2 \delta\phi = -4\pi G(\rho_{\rm tot} + \delta\rho_\infty). \tag{298}$$

Using the Jeans wavelength[15] (Zeldovich & Novikov 1971):

$$k_{\rm J}^2 = \frac{4\pi G\rho_\infty}{a_\infty^2}, \tag{299}$$

we can cast Eqs. (296-297) in the following form of an equation in the perturbations produced by the moving gravitational center:

$$\Box^2\delta + k_{\rm J}^2 a_\infty^2 \delta = -4\pi G\rho_{\rm tot} + \xi_1 \pi R_{\rm G}^2 v_\infty \frac{\partial}{\partial t}\delta(R - v_\infty t). \tag{300}$$

To simplify Eq. (300), we can center the center the coordinate system in the moving object and neglect self-gravity of the gas ($k_{\rm J}$). Then, (300) can be written as:

$$(a_\infty^2 - v_\infty^2)\nabla^2\delta = 4\pi GM\delta(R). \tag{301}$$

The solution is (Lipunov 1992):

$$\delta = \frac{R_{\rm G} v_\infty}{a_\infty R\left(1 - \frac{v_\infty^2}{a_\infty^2 \sin^2\theta_{\rm sh}}\right)^{1/2}}. \tag{302}$$

If we introduce the Mach number $M_{\rm M} = v_\infty/a_\infty$, we see from Eq. (302) that there is a singularity at the surface of the cone described by:

$$\sin\theta_{\rm sh} = \frac{1}{M_{\rm M}}. \tag{303}$$

This means that the solution (302) is not valid close to the cone, which implies that a shock wave is formed, with a form of a cone with opening angle $\theta_{\rm sh}$ (see Fig. 30). This shock is called a "bow shock".

A force due to dynamical friction opposes to the motion of the compact object, slowing it down. Such a force is the result that the density in the background matter in the wake is higher than in front of the moving center. The dynamic friction force is:

$$F_{\rm fr} = \pi R_{\rm G}^2 \rho_\infty v_\infty. \tag{304}$$

Numerical simulations show that the accretion flow pattern is complicated and dependent on the efficiency of the gas cooling mechanism. The simulations also show the formation of a frontal shock at a distance $\sim R_{\rm G}$. This region is prone to suffer Reyleight-Taylor instabilities (e.g. Araudo et al. 2009). In Fig. 31 we show a 3D Smooth Particle Hydrodynamic (SPH) simulation of a moving black hole in a stellar wind.

The temperature in the wake of the the shock is (Lang 1999):

$$T_{\rm sh} = \frac{m_p v_\infty^2}{6k} \approx 2.5 \times 10^5 \left(\frac{v_\infty}{10^7 \text{ cm s}^{-1}}\right)^2 \text{ K}, \tag{305}$$

where k is the Boltzmann's constant.

A realistic study of the accretion regimes onto moving objects requires extensive numerical simulations.

[15]The Jeans wavelength $k_{\rm J}$ is defined such that any small sinusoidal density disturbance with a wavelength exceeding $2\pi/k_{\rm J}$ will be gravitationally unstable. The Jeans critical mass is usually defined as the density times the cubic of the length. Higher masses than the Jeans mass start to condense gravitationally.

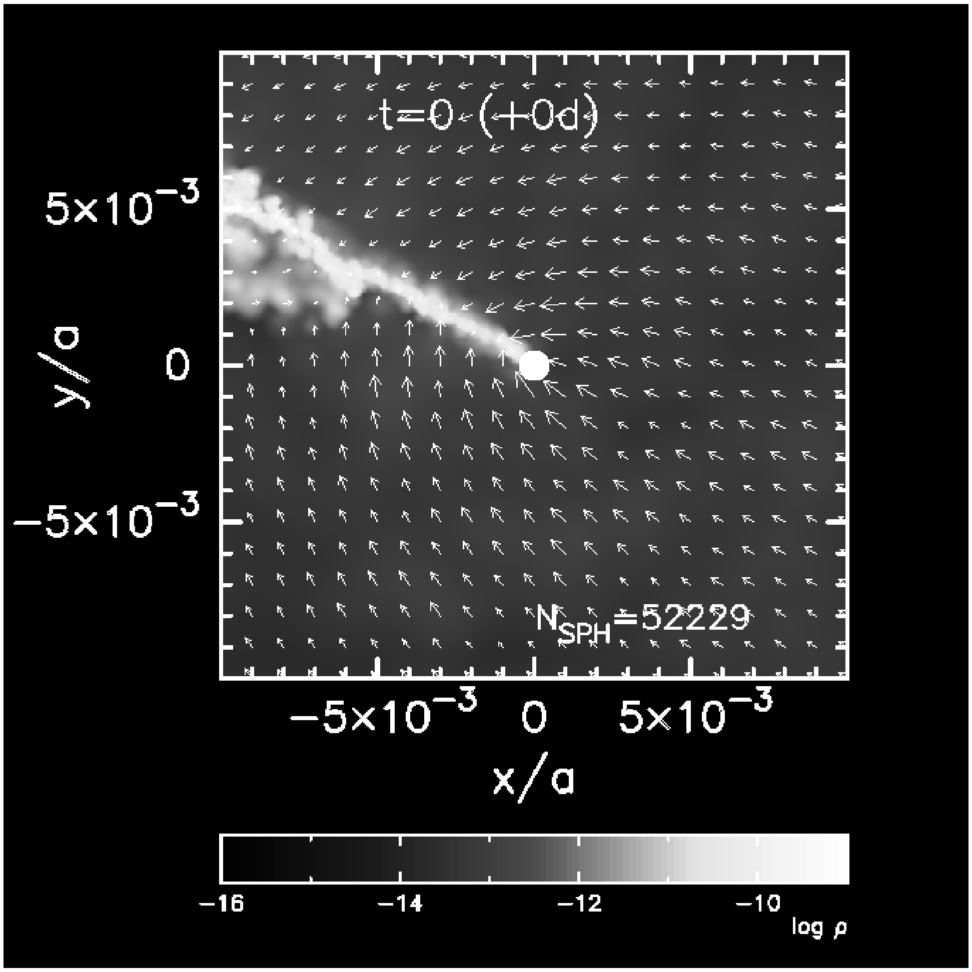

Figure 31. 3D-SPH simulation of the accretion onto a black hole moving through a stellar wind. The arrows indicete the direction of the flow. From Okazaki et al. (2009).

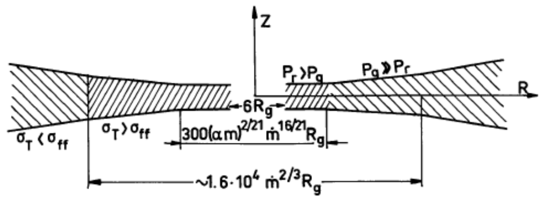

Figure 32. Accretion disk in the Shakura-Sunyaev model, with regions of different physical conditions. From Shakura & Sunyaev (1973).

25.3. Disk accretion

In most realistic astrophysical situations the matter captured by a gravitational field will have a total non-zero angular momentum. The accretion of matter with angular momentum onto a black hole leads to the formation of an accretion disk. The main difficulty in the formulation of a consistent theory of accretion disks lies in the lack of knowledge on the nature of turbulence in the disk and, therefore, in the estimate of the dynamic viscosity.

We shall consider steady state accretion disk where the accretion rate is considered as an external parameter and the characterization of the turbulence is provided by a unique parameter: the so-called α parameter, introduced by Shakura (1972) and Shakura & Sunyaev (1973).

We shall start with the following simplifying assumptions: 1) the disk is thin, i.e. its characteristic scale in the z-axis is $H << R$ (see Fig. 32), 2) the matter in the disk is in hydrostatic equilibrium in the z-axis, 3) self-gravitation of the disk can be neglected. Condition 2) can be expressed as:

$$\frac{1}{\rho}\frac{dP}{dz} = -\frac{GM}{R^3}z. \tag{306}$$

If $a_{\rm s}$ is the sound speed, $H = \Delta z$ is the half-thickness of the disk, and $P = \rho a_{\rm s}^2$, we can re-write Eq. (306) in the following way:

$$a_{\rm s} = \omega_{\rm K} H, \tag{307}$$

where

$$\omega_{\rm K} = \sqrt{\frac{GM}{R^3}} \tag{308}$$

is the Keplerian angular velocity. Assuming circular orbits:

$$v_\phi = \sqrt{\frac{GM}{R}} = \omega_{\rm K} R. \tag{309}$$

Form Eqs.(308) and (309), it follows

$$\frac{a_{\rm s}}{v_\phi} \approx \frac{H}{R}. \tag{310}$$

Notice that since the particles move into Keplerian orbits there is no pressure gradient along R. The transport of angular momentum along the disk is associated with the moment of viscous forces:

$$\dot{M}\frac{d\omega_{\rm K}}{dR}R^2 = 2\pi\frac{d}{dR}W_{r\phi}R^2. \tag{311}$$

Here, $W_{r\phi}$ is the component of the viscous stress in the disk:

$$W_{r\phi} = -2\eta HR\frac{\partial\omega_{\rm K}}{\partial R}. \tag{312}$$

The parameter η is the dynamic viscosity averaged over the z-coordinate. Notice that for a rigid body $\partial\omega_{\rm K}/\partial R = 0$ and the viscous stress vanishes.

The viscosity of isotropic turbulence is (Landau & Lifshitz 2002): $\eta = (1/3)\rho v_{\rm t} l_{\rm t}$, where $v_{\rm t}$ and $l_{\rm t}$ are the characteristic velocity and scale of the turbulence, respectively. Shakura (1972) introduced the following expression to characterize the viscous fluid:

$$v_{\rm t} l_{\rm t} = \alpha a_{\rm s} H, \tag{313}$$

where α is the viscosity parameter. Since $v_{\rm t} < a_{\rm s}$[16] and $l_{\rm t} < H$, then $\alpha \leq 1$.

Equation (311) can be integrated yielding:

$$W_{r\phi} = -\frac{\dot{M}}{2\pi}\omega_{\rm K}\left[1-\left(\frac{R_{\rm d}}{R}\right)^{1/2}\right] + W_{r\phi}({\rm in}), \tag{314}$$

where $R_{\rm d}$ is the radius of the inner edge of the disk and $W_{r\phi}({\rm in})$ is the component of the tensor of viscous stress evaluated at $R = R_{\rm d}$. As we have seen before, for a Schwarzschild black hole the last stable orbit is at $R_{\rm d} = 3R_{\rm Schw}$. For this last orbit, we can take:

$$W_{r\phi}({\rm in}) = 0.$$

The continuity equation can be written as:

$$\dot{M} = 2\pi\rho(2H)Rv_R, \tag{315}$$

with v_R the radial velocity of the matter in the accretion disk.

Far from the inner edge we have:

$$W_{r\phi} \approx -\frac{\dot{M}}{2\pi}\omega_{\rm K} \approx 3\eta H\omega_{\rm K} = \alpha PH. \tag{316}$$

From this equation and eq. (315) it can be obtained that

$$\frac{v_R}{v_\phi} \approx \alpha\left(\frac{H}{R}\right)^2. \tag{317}$$

[16] Turbulence is quickly attenuated in supersonic flows.

The transport of angular momentum in the disk results in the generation of heat. We can express the heat produced per unit surface area of the disk per unit of time on each side as:

$$Q^{+} = -\frac{1}{2} W_{r\phi} R \frac{d\omega}{dR} = \frac{3}{4}\omega W_{r\phi}. \tag{318}$$

This energy is carried away in the form of thermal radiation:

$$Q^{-} = \sigma_{\rm SB} T^4, \tag{319}$$

where $\sigma_{\rm SB}$ is the Stephan-Boltzmann's constant and we have assumed that the disk radiates as a black body. In the steady state $Q^{+} = Q^{-}$. If $Q^{+} > Q^{-}$ the disk becomes thermally unstable. The full set of equations that determine the disk accretion are:

1. $\omega = \omega_{\rm K} = \left(\frac{GM}{R^3}\right)^{1/2}$ (Kepler's law).

2. $\dot{M} = -2\pi\Sigma v_R R$ (continuity equation).

3. $W_{r\phi} = -\frac{\dot{M}}{2\pi}\omega_{\rm K}\left[1 - \left(\frac{R_{\rm d}}{R}\right)^{1/2}\right] + W_{r\phi}(\text{in})$ (variation of angular momentum).

4. $P = \frac{\Sigma\omega^2 H}{6}$ (hydrostatic equilibrium).

5. $W_{r\phi} = \alpha P H$ (viscous tensor).

6. $Q^{+} = -\frac{1}{2} W_{r\phi} R \frac{d\omega}{dR}$ (energy release).

7. $Q^{-} = \sigma_{\rm SB} T^4$ (losses by radiation).

8. $P = \frac{3}{2}\rho R_{\rm u}(T_e + T_i) + \frac{\epsilon}{3}$ (equation of state, with ϵ the energy density).

9. $\sigma[\text{cm}^2] = \sigma_{\rm T} + \sigma_{\rm ff} \approx 6.65 \times 10^{-25}\text{n} + \frac{1.8\times 10^{-25}}{\text{T}^{7/2}}$ (absorption cross section).

This is a system of 9 equations with 9 functions of R as solution. The solutions where found by Shakura & Sunyaev (1973). For fixed values of M and $\dot{M}$, the disk can be into three different regions:

- An outer region (large R) where the gas pressure dominates over radiation pressure and opacity is controlled by free-free absroption.

- A middle region (smaller R) where the gas pressure dominates over radiation pressure but the opacity is due to electron scattering.

- An inner region (very small R) where radiation pressure dominates over gas pressure and the opacity is also due to electron scattering.

Luminosity and spectrum of standard accretion disks

The energy carried with the radiation released in a ring of thickness dR of the accretion disk is:

$$dL(R) = 2Q^{+}2\pi R dR, \tag{320}$$

where the initial factor 2 is due to the two faces of the disk. This equation can be written as:

$$dL(R) = \frac{3}{2}\dot{M}\frac{d}{dR}\frac{GM}{R^2}\left(1-\sqrt{\frac{R_{\rm d}}{R}}\right)dR. \tag{321}$$

The power $dL(R)$ corresponds to the work done by the gravitational field. Approximately half of this power is transformed into kinetic energy of the matter moving along ϕ and the other half is transformed into heat:

$$dL_{\rm gr} = \dot{M}\frac{d}{dr}\left(-\frac{GM}{2R}\right)dR = \frac{1}{2}\dot{M}\frac{GM}{R^2}dR. \tag{322}$$

Comparing Eqs. (321) and (322) we see that at large distances from the inner edge the dissipation rate is three times larger than the gravitational energy release. This is because of th viscous effects, the half of the gravitational energy is radiated from the disk *globally* but not *locally*. The total luminosity of the disk is:

$$L_{\rm d} = \int_{R_{\rm d}}^{\infty}\frac{dL(R)}{dR}dR = \frac{\dot{M}GM}{2R}. \tag{323}$$

Adopting $R_{\rm d} = 3R_{\rm Schw}$ and dividing by $\dot{M}c^2$ we get the efficiency of energy release in the disk accretion process: ~ 8 %. For a Kerr black hole, where $R_{\rm d} = R_{\rm g}$, the efficiency reaches ~ 42 %.

Using Eq. (321) we get:

$$Q^{+} = \frac{3}{8\pi}\frac{\dot{M}GM}{R^3}, \tag{324}$$

for $R >> R_{\rm d}$. Through the energy energy balance equation $Q^{+} = Q^{-} = \sigma_{\rm SB}T^4$ we can obtain the temperature distribution along the radial direction in the disk:

$$T(R) = \left(\frac{3}{8\pi\sigma_{\rm SB}}\dot{M}\frac{GM}{R^3}\right)^{1/4} \propto R^{-3/4}. \tag{325}$$

The total spectrum is the result of the superposition of the blackbody emission from each ring of temperature $T(R)$:

$$I_\nu = 2\pi\int_{R_{\rm d}}^{R_{\rm out}} B_\nu[T(R)]RdR, \tag{326}$$

with

$$B_\nu(T) = \frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^3\frac{x^3}{e^x-1}; \quad x \equiv \frac{hv}{kT}, \tag{327}$$

the Planck's function. The result of the integration for $R_{\rm out} >> R_{\rm d}$ is:

$$I_\nu = \frac{16\pi^2 R_{\rm d}^2}{c^2}\left(\frac{kT_{\rm d}}{h}\right)^{8/3} h\nu^{1/3}. \tag{328}$$

The typical temperature can be obtained from:

$$\frac{1}{2}\dot{M}\frac{GM}{R_{\rm d}} = 2\pi R_{\rm d}^2 \sigma_{\rm SB} T^4 \tag{329}$$

The result is:

$$T = \left(\frac{\dot{M}GM}{4\pi R_{\rm d}^3 \sigma_{\rm SB}}\right)^{1/4}. \tag{330}$$

This yields temperatures of $\sim 10^7$ K for stellar mass black holes in binary systems ($\dot{M} \sim 10^{18}$ g s^{-1}).

25.4. Advection-dominated accretion flows

Shapiro, Lightman, and Eardly (1976) found a self-consistent solution for the hydrodynamical equations of an accreting flow onto a compact object, including both rotation and viscosity. This solution has the characteristic that the plasma has two-temperatures. The ion temperature ($T \sim 10^{12}$ K) is much higher than the electron temperature ($T \sim 10^9$ K). The plasma is optically thin and the radiation has a power-law spectrum in X-rays, consistent with what is observed in sources like Cygnus X-1. However, the solution is thermally unstable.

Thermally stable solutions were found by Begelman and Meier (1982) in a super-Eddington accretion regime (the disk results optically thick) and by Narayan and Yi (1994a,b, 1995a,b). The latter solution corresponds to sub-Eddington accretion of a low-density gas. The energy released by viscosity is stored in the plasma, which is advected and swallowed by the black hole. The plasma is optically thin and with two temperatures. This type of solution describes what is known as advection-dominated accretion flows (ADAFs).

The different accretion regimes in an ADAF are determined by the parameter f defined as:

$$f = \frac{Q^+ - Q^-}{Q^+} \equiv \frac{Q_{\rm adv}}{Q^+}, \tag{331}$$

i.e. as the ratio between the advected energy and the energy released through viscosity. Different values of f correspond to different types of accretion.

- $f \ll 1$: in this case $Q^+ \approx Q^- \gg Q_{\rm adv}$ and all the energy released by viscosity is radiated. This regime corresponds to thin disks and two-temperature solutions such as that of Shapiro et al. (1976).

- $f \approx 1$: here $Q_{\rm adv} \approx Q^+ \gg Q^-$, the cooling is negligible and the flow is ADAF-like.

- $|f| \gg 1$: corresponds to $-Q_{\rm adv} \approx Q^- \gg Q^+$. The situation is like in the Bondi-Hoyle regime.

If we consider spherical coordinates $(R,\ \theta,\ \phi)$ the three components of the Euler equation and and the energy conservation can be written as:

$$\begin{aligned}\rho\left(v_R\frac{\partial v_R}{\partial R}-\frac{v_\phi^2}{R}\right) &= -\frac{GM\rho}{R^2}-\frac{\partial p}{\partial R}+\frac{\partial}{\partial R}\left[2\nu\rho\frac{\partial v_R}{\partial R}-\frac{2}{3}\nu\rho\left(\frac{2v_R}{R}+\frac{\partial v_R}{\partial R}\right)\right]\\ &+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\nu\rho}{R}\frac{\partial v_R}{\partial\theta}\right)+\frac{\nu\rho}{R}\left[4R\frac{\partial}{\partial R}\left(\frac{v_R}{R}\right)+\frac{\cot\theta}{R}\frac{\partial v_R}{\partial\theta}\right],\end{aligned}$$

$$\begin{aligned}\rho\left(-\frac{\cot\theta}{R}v_\phi^2\right) &= -\frac{1}{R}\frac{\partial p}{\partial\theta}+\frac{\partial}{\partial R}\left(\frac{\nu\rho}{R}\frac{\partial v_R}{\partial\theta}\right)\\ &+\frac{1}{R}\frac{\partial}{\partial\theta}\left[\frac{2\nu\rho}{R}v_R-\frac{2\nu\rho}{3}\left(\frac{2v_R}{R}+\frac{\partial v_R}{\partial R}\right)\right]+\frac{3\nu\rho}{R^2}\frac{\partial v_R}{\partial\theta},\end{aligned}$$

$$\begin{aligned}\rho\left(v_R\frac{\partial v_\phi}{\partial R}+\frac{v_\phi v_R}{R}\right) &= \frac{\partial}{\partial R}\left[\nu\rho R\frac{\partial}{\partial R}\left(\frac{v_\phi}{R}\right)\right]+\frac{1}{R}\frac{\partial}{\partial\theta}\left[\frac{\nu\rho\sin\theta}{R}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right)\right]\\ &+\frac{\nu\rho}{R}\left[3R\frac{\partial}{\partial R}\left(\frac{v_\phi}{R}\right)+\frac{2\cot\theta\sin\theta}{R}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right)\right],\end{aligned}$$

$$\begin{aligned}\rho\left(v_R\frac{\partial\varepsilon}{\partial R}-\frac{p}{\rho^2}v_R\frac{\partial\rho}{\partial R}\right) &= -\frac{2f\nu\rho}{3}\left[\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2v_R\right)\right]^2+2f\nu\rho+\left\{\left(\frac{\partial v_R}{\partial R}\right)^2+2\left(\frac{v_R}{R}\right)^2\right.\\ &\left.+\frac{1}{2}\left(\frac{1}{R}\frac{\partial v_R}{\partial\theta}\right)^2+\frac{1}{2}\left[R\frac{\partial}{\partial R}\left(\frac{v_R}{R}\right)\right]^2+\frac{1}{2}\left[\frac{\sin\theta}{R}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right)\right]^2\right\}.\end{aligned}$$

Narayan and Yi (1995a) found the following self-similar solutions for these equations:

$$v_R = R\Omega_{\rm K}(R)v\left(\theta\right), \tag{332}$$

$$v_\theta = 0, \tag{333}$$

$$v_\phi = R\Omega_{\rm K}(R)\Omega(\theta), \tag{334}$$

$$c_s = R\Omega_{\rm K}(R)c_s(\theta), \tag{335}$$

$$\rho = R^{-3/2}\rho(\theta). \tag{336}$$

Two-temperature ADAF models are based in a series of hypothesis about the thermodynamics of the accretion gas:

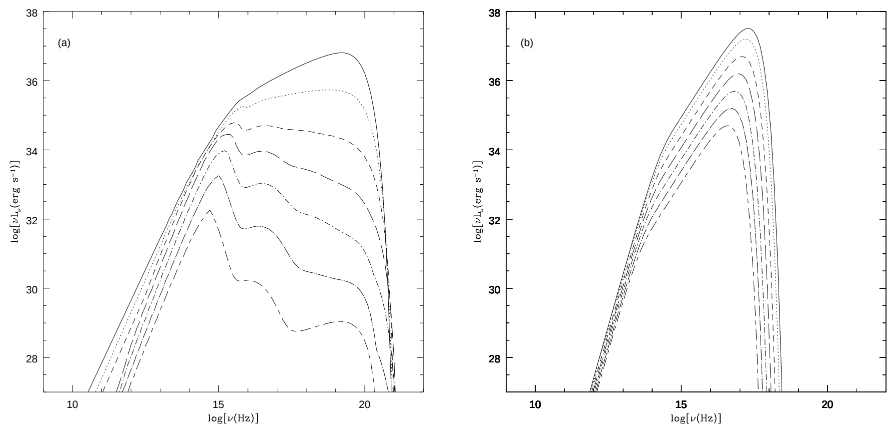

Figure 33. Left: ADAF spectra from a 10 $M_\odot$ black hole and different accretion rates. *Right*: Thin disk spectra for the same accretion rates and black hole.

- The total pressure has contributions from both the gas and the magnetic field:

$$p = p_{\rm g} + p_{\rm m}. \tag{337}$$

The magnetic pressure is:

$$p_{\rm m} = \frac{B^2}{8\pi}, \tag{338}$$

and the gas pressure, in the case of an ideal gas of density n and temperature T,

$$p_{\rm g} = nkT. \tag{339}$$

The firs hypothesis is that the magnetic pressure is a fixed fraction of the gas pressure:

$$p_{\rm m} = (1-\beta)\, p, \qquad\qquad p_{\rm g} = \beta p. \tag{340}$$

The value $\beta = 0.5$ corresponds to strict equipartition.

- A second hypothesis is that the temperature of ions and temperature of electrons are different. Then, the gas pressure becoms:

$$p_{\rm g} = \beta\rho c_s^2 = \frac{\rho}{\mu_i m_H} kT_i + \frac{\rho}{\mu_e m_H} kT_e, \tag{341}$$

where m_H is the hydrogen mass and $\mu_{i,e}$ the molecular weight of ions and electrons, respectively.

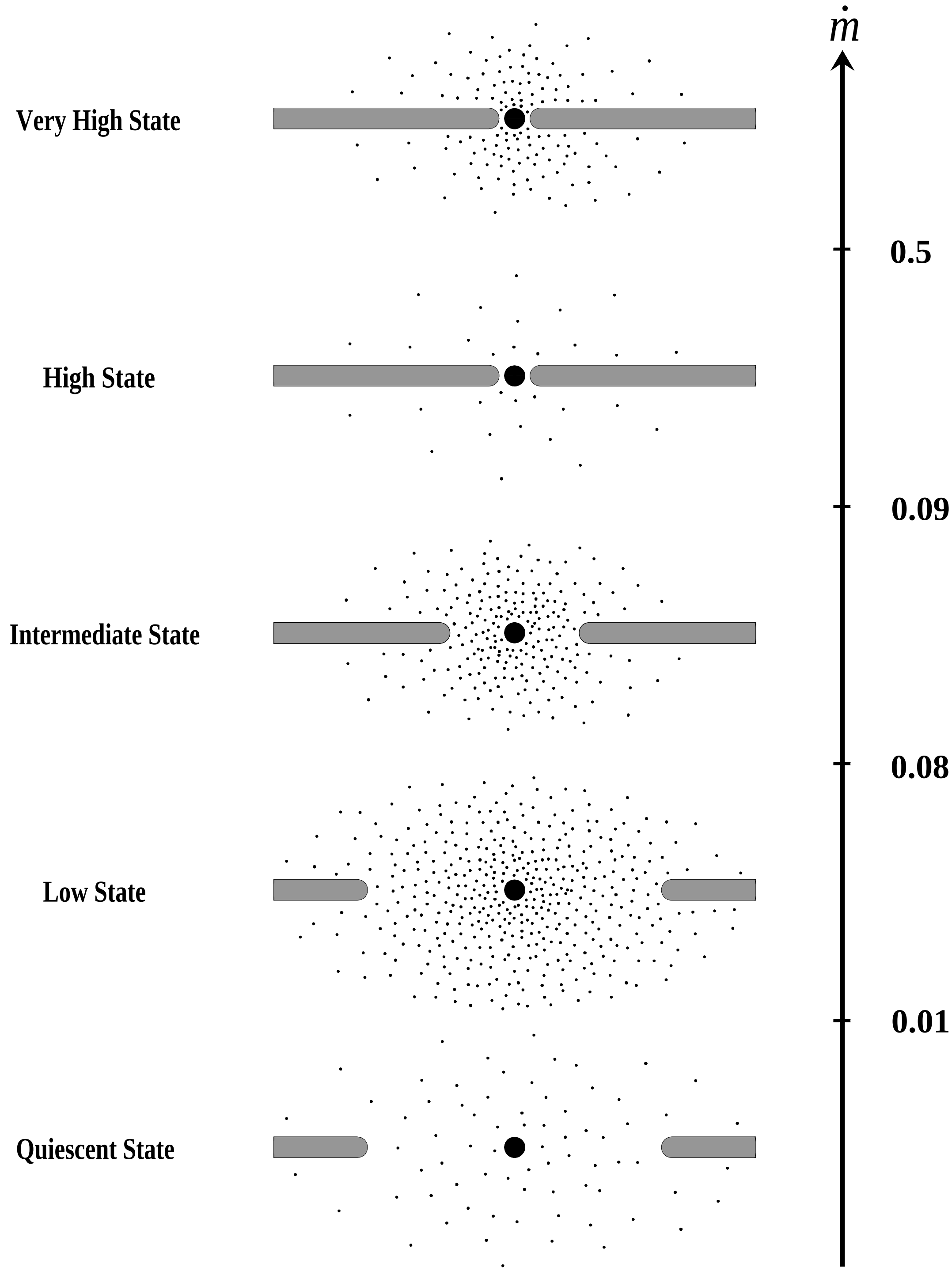

Figure 34. Geometry of the flux in different spectral states as a function of the accretion rate normalized in Eddington units (Esin et al. 1997).

The radiation pressure is not considered since in flows dominated by advection is ussually negligible.

- There is a preferential heating of ions. Because of the large difference is mass, it is assumed that the energy released by viscosity is transferred to ions and just a small fraction $\delta \ll 1$ goes to electrons. Usually, it is assumed $\delta \sim 10^{-3} \sim m_e/m_p$. In such a case, the result will be $T_i \gg T_e$, where typically $T_i \sim 10^{12}$ K and $T_e \sim 10^9$ K. Even if both types of particles receive the same amount of energy, electrons will cool more efficiently, leading to $T_i \gg T_e$ in any case.

- ADAF models assume that there is no thermal coupling between iones and electrons, and the only relevant interaction is Coulombian.

- The resulting spectrum of the ADAF will result from the operation of the different cooling mechanisms. For electrons the most relevant mechanisms are synchrotron radiation, Bremsstrahlung, and inverse Compton scattering:

$$Q_e^- = Q_{\rm Br}^- + Q_{\rm synchr}^- + Q_{\rm IC}^-. \tag{342}$$

Photons produced by Bremsstrahlung and synchrotron process can be upscattered by electrons, in addition to those coming from external fields. Then, $Q_{\rm IC}^-$ can be written as:

$$Q_{\rm IC}^- = Q_{\rm IC,Br}^- + Q_{\rm IC,synchr}^- + Q_{\rm IC,ext}^-. \tag{343}$$

In the steady state the energy gained by the ions through the viscous heating must be equal to the energy transferred to the electrons plus the advected energy:

$$Q^+ = Q_{\rm adv} + Q_{ie} = fQ^+ + Q_{ie}. \tag{344}$$

This assumes that the ions have no radiative losses.

In Figure 35 we show a typical ADAF spectrum.

Recently, Romero, Vieyro and Vila (2010) have considered the effect of non-thermal particle populations in the central region of ADAF-like systems. In such a case relativistic protons cool by synchrotron, photo-pair, photo-meson and inelastic proton-proton interactions. The full non-thermal spectrum can be quite complicated. An example us shown in Figure 36 and the application to the classic source Cygnus X-1 in Figure 36.

25.5. Accretion in binary systems

Zel'dovich (1964) was the first to suggest that black holes might be detected by the emission due to accretion from a companion star in a binary system. There are two basic process through which the black hole can capture the material from the star: 1) production of gas jets by overflow of the Roche lobe in the case of a normal star, 2) gravitational capture of matter forming the stellar wind of the star. These winds are particularly strong in the case of massive, early-type stars. In what follows we shall briefly review these modes of accretion (see Frank, Kind and Raine 1992 for details).

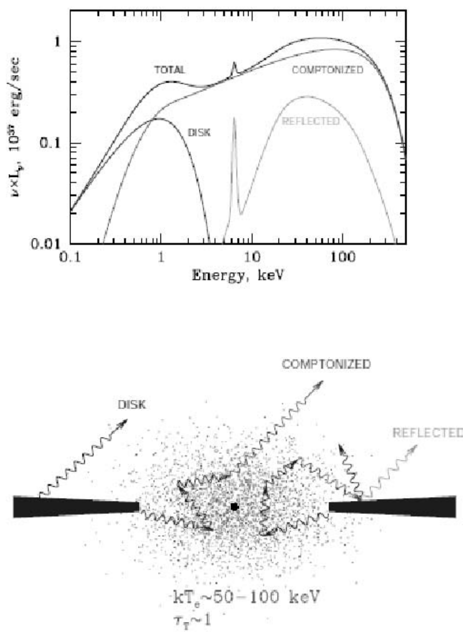

Figure 35. Different contribution to the total spectrum of an ADAF.

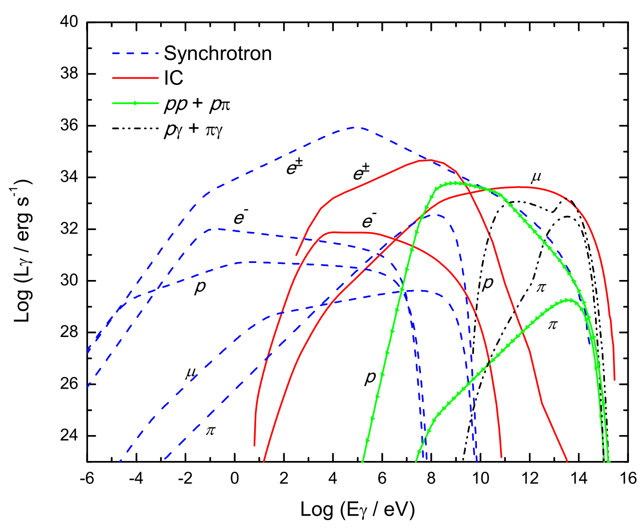

Figure 36. Non-thermal contributions to an ADAF/corona model with a relativistic proton-to-electron energy density ration of 100.

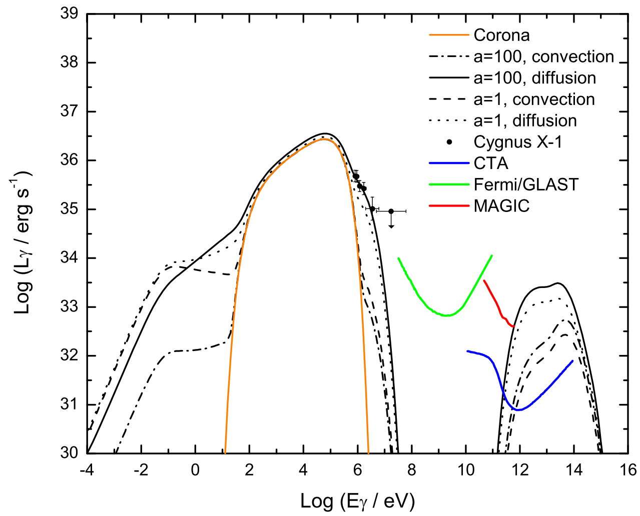

Figure 37. ADAF/corona model with non-thermal contribution for Cygnus X-1. Different cases are considered, with different proton content.

Overflow of the Roche lobe

Let us consider a black hole of mass $M_\bullet$ and a star of mass M_* in a circular orbit. In the plane of the orbit, the effective potential resulting from the gravitational and centrifugal forces is (e.g. Lipunov 1992):

$$\Phi = -\frac{GM_*}{R_1} - \frac{GM_\bullet}{R_2} + \frac{\Omega^2(x^2+y^2)}{2}, \tag{345}$$

where Ω is the angular velocity associated to the radius of the orbit a:

$$a = \left[\frac{G(M_\bullet + M_*)}{\Omega^2}\right]^{1/3}, \tag{346}$$

and $R_1^2 = x_1^2 + y_1^2$, $R_2^2 = x_2^2 + y_2^2$ indicate the distances of a particle to M_* and $M_\bullet$, respectively.

The total energy of the particle is:

$$\Phi + \frac{1}{2}v^2 = E_0. \tag{347}$$

For low energies, the particle, if emitted by the star, will fall back to it. At the turning point its velocity will be $v = 0$. In such a case, $\Phi = E_0$. This equation defines equipotential surfaces (Hill's surfaces) that limit the motion of particles of energy E_0. At a certain energy $E_{\rm R} = \Phi_{\rm R}$, the Hill's surfaces of both masses come in contact and form the so-called Roche lobe. The point of contact is the inner Lagrangian point where the resultant of all forces equals zero:

$$\frac{d\Phi}{dx} = 0. \tag{348}$$

When the star fills its Roche lobe it injects matter over the black hole through the inner Lagrangian point. There, a particle can go from one lobe to the other without losing energy. The specific angular momentum l_Ω of the injected gas is related to the angular momentum of the orbital motion:

$$l_\Omega \approx \Omega a^2. \tag{349}$$

The thickness of the gas jet crossing the Lagrange point is $\sim 0.1\ R_*$. The flow rate is determined by the state of evolution of the star and the ratio $q = M_\bullet/M_*$. For $q \ll 1$, the flow occurs on a time scale

$$\tau = \frac{GM_*}{R_* L_*}, \tag{350}$$

where L_* is the total luminosity of the star. Then, the matter flow rate is:

$$\dot{M} = \frac{M_*}{\tau}. \tag{351}$$

The jet of gas collides with the outer part of the disk. Since the size of the disk is of the order of the size of the Roche lobe, we can write the time the gas takes to move along the whole disk as:

$$t_R \approx \frac{R_{\rm out}}{v_R} \approx \frac{a}{v_r} \approx \frac{a}{\alpha v_\phi (H/R)^2} \approx\approx \frac{T}{2\pi\alpha}\left(\frac{R}{H}\right)^2, \tag{352}$$

where we have used the standard thin disk solution and T is the orbital period. Since for thin disks $H \ll R$ and $\alpha < 1$, the time scale of radial motion of matter is longer than the orbital period:

$$\frac{t_R}{T} \gg 1. \tag{353}$$

Accretion from stellar winds

The stellar wind of early-type stars is both strong ($10^6 - 10^{-4}\ M_\odot\ {\rm yr}^{-1}$) and supersonic. The wind velocity is:

$$v_{\rm w} \approx v_{\rm esc} \approx \sqrt{\frac{2GM_*}{R_*}}. \tag{354}$$

For typical values this velocity will be of a few thousands km s^{-1}, a highly supersonic value since for the interstellar medium $c_{\rm s} \sim 10$ km s^{-1}. If the orbital velocity of the black hole in the high-mass binary system is $v_\bullet$, the wind moves forming an angle $\beta = \tan^{-1}(v_\bullet/v_{\rm w})$ with the symmetry axis of the bow shock. The relative velocity is:

$$v_{\rm rel} = (v_\bullet^2 + v_{\rm w}^2)^{1/2}. \tag{355}$$

Gravitational capture of wind particles will occur within a cylindrical region of radius

$$R_{\rm G} \sim \frac{2GM_\bullet}{v_{\rm rel}^2}. \tag{356}$$

Considering $v_{\rm w} \gg v_\bullet$, we have $\beta = 0$ and $v_{\rm rel} = v_{\rm w}$. Then, the fraction of the stellar wind captured by the black hole is given by the ratio of the mass flux into the accretion cylinder and the total mass loss rate of the star $\dot{M}_*$:

$$\frac{\dot{M}}{\dot{M}_*} \approx \frac{\pi R_{\rm G}^2 v_{\rm w}(a)}{4\pi a^2 v_{\rm w}(a)} = \frac{G^2 M_\bullet^2}{a^2 v_{\rm w}^4(a)}, \tag{357}$$

where a is the binary separation. Using the expression for $R_{\rm G}$:

$$\frac{\dot{M}}{\dot{M}_*} \approx \frac{1}{4}\left(\frac{M_\bullet}{M_*}\right)^2 \left(\frac{R_*}{a}\right)^2. \tag{358}$$

For typical values this implies accretion rates of $10^{-4} - 10^{-3}\ M_*$. Thus, the accretion from the stellar wind is far less efficient than the accretion by Roche lobe overflow, where the efficiency is almost 1. However, since the mass loss rate of the massive stars is so high, the resulting luminosity is observable. Typically:

$$L_{\rm acc} = \eta \dot{M} c^2 \sim 10^{37} \left(\frac{\dot{M}}{10^{-4}\ \dot{M}_*}\right) \left(\frac{\dot{M}_*}{10^{-5}\ M_\odot\ {\rm yr}^{-1}}\right)\ \ {\rm erg\ s}^{-1}. \tag{359}$$

Although the wind has no angular momentum if the star is a slow rotator, a disk can be formed around the black hole because of the orbital motion. The associated specific angular momentum is then

$$l = \frac{1}{4} R_{\rm disk}^2 \left(\frac{v_\bullet}{a}\right), \tag{360}$$

where $R_{\rm disk}$ is the radius of accretion disk. For short orbital periods (small a) significant disks can be formed.

26. Jets

Jets are collimated flows of particles and electromagnetic fields. They are observed in a wide variety of astrophysical systems, from protostars to Active Galactic Nuclei (AGNs). Astrophysical jets seems to be associated with accretion onto a compact central object. The most remarkable property of jets is that their length exceeds the size of the compact object by many orders of magnitude. For instance, in AGNs the jets are generated in a region of no more than 100 gravitational radii of the central black hole ($\sim 10^{15}$ cm), and propagate up to distances of $\sim 10^{24}$ cm, well into the intergalactic medium. Along the jets, the specific volume of plasma increases enormously and the corresponding adiabatic losses, in combination with various radiative losses, ensure that the particles lose essentially all their "thermal" energy very quickly. Yet, high-resolution radio and X-ray observations show that the jet brightness does not decline so rapidly. This suggests that most of the jet energy is in a different form and that the observed emission is the results of its slow dissipation.

One possibility is that astrophysical jets are supersonic, kinetic energy-dominated flows. A number of factors make the idea very attractive. First, such flows do not require external support in order to preserve their collimation.

Second, they are stable and can propagate up to large distances without significant energy losses, in an essentially ballistic regime. Third, when they interact with the external medium the result is shock formation. These shocks dissipate kinetic energy locally and thus can produce bright compact emission sites, like the knots and hot spots of observed in astrophysical jets.

26.1. Acceleration

The current magnetic acceleration model for jets assumes that most of the jet Poynting flux is first converted into bulk kinetic energy and then dissipated at shocks (shocks in highly magnetized plasma are inefficient). The generation of the jets is the result of magneto-centrifugal forces in the inner part of the accretion disk (Blandford and Payne 1982). The rotating magnetic field lines anchored in the disk can drive material from the through centrifugal forces if the angle with the symmetry axis of the accretion disk is $\geq 60°$. This material might be extracted from the tenuous region right below and above the disk surface, the disk atmosphere or corona, where, because of matter dilution the magnetic pressure becomes larger than the gas pressure. Provided the magnetic field has the proper geometry, the material, of density ρ, will be *pushed* along the magnetic lines due to the high conductivity of the plasma, suffering a force along the field lines and accelerating up to a velocity of the order of the Alfvén velocity $v_{\rm A} = B/(4\pi\rho)^{1/2}$, which in the disk surface is well above the sound speed $c_{\rm s}$. At this point, the dynamics is again dominated by matter, magnetic lines get bent, and magneto-centrifugal acceleration is quenched, but the material has already reached a significant velocity. This outflow can be collimated by the lateral pressure of matter and magnetic fields of the corona (the nozzle) forming a jet. The origin of the energy of the jet is the accretion disk, not the magnetic field. The matter loading, the final velocity, and the collimation degree of the jet depend strongly on the conditions in the disk atmosphere.

The different regions in a magnetically accelerated outflow are indicated in Figure 38 (Spruit 2010). The Alfvén radius is the point where the flow speed equals the Alfvén speed.

The acceleration of the jet depends on the inclination of the field lines: there is a net upward force along the field lines only if they are inclined outward at a sufficient angle. Field lines more parallel to the axis do not accelerate a flow. The conditions for collimation and acceleration thus conflict somewhat with each other. Explanation of the very high degree of collimation observed in some jets requires additional arguments (e.g. external confinement).

At the Alfvén radius the flow has reached a significant fraction of its terminal velocity. The field lines start lagging behind, with the consequence that they get 'wound up' into a spiral. Beyond the Alfvén radius, the rotation rate of the flow gradually vanishes by the tendency to conserve angular momentum, as the flow continues to expand away from the axis (see Figure 39).

In Figure 40 we show a disk+jet system with the associated magnetic field. In this case the magnetic tower formed by the field helps to confine the outflow.

At the beginning, the jet dynamics is dominated by the magnetic field (B). In such a case, jet luminosity, assuming ideal plasma conditions, is given by the Poynting flux:

$$L_{\rm j} \approx \mathbf{S} = c/8\pi A_{\rm j}\,|\mathbf{E}\times\mathbf{B}| = c/8\pi|(\mathbf{v}\times {\rm B})\times {\rm B}|\,, \tag{361}$$

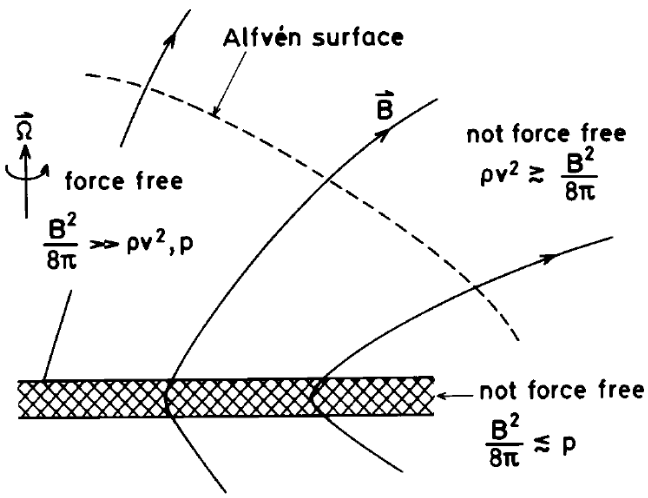

Figure 38. Different regions in a magnetically accelerated outflow from an accretion disk (from Spruit 2010).

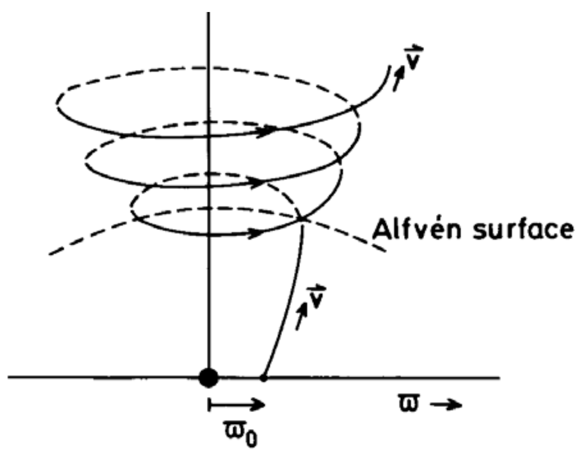

Figure 39. Behavior field lines. Beyond the Alvén radius the field is twisted into a spiral (from Spruit 2010).

where $A_{\rm j}$ is the jet section. When the jet is dominated by matter, the luminosity becomes

$$L_{\rm j} \approx (\Gamma_{\rm j} - 1)\, A_{\rm j}\, \rho_{\rm j}\, c^2\,, \tag{362}$$

where $\rho_{\rm j}$ is the mass density of the jet, $\Gamma_{\rm j} = 1/\sqrt{1-(v_{\rm j}/c)^2}$ is the bulk Lorentz factor, and $v_{\rm j}$ the jet's velocity.

In case the magnetic field has not a dominant role, the jet evolution can be described using hydrodynamical (HD) equations. Otherwise, a magnetohydrodynamical (MHD) treatment, or in some extreme cases a pure electromagnetic

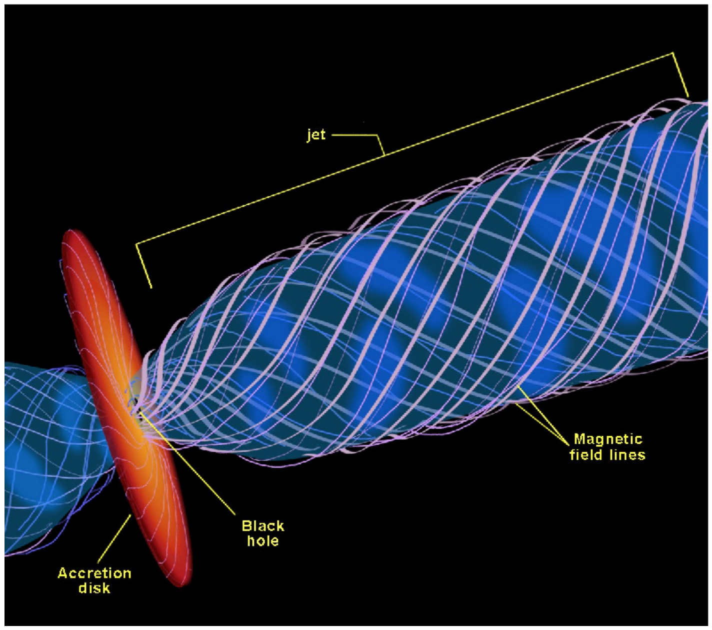

Figure 40. Jet, accretion disk, and black hole system. The magnetic field geometry is shown.

(EM) one, is needed. Even when an MHD approach is suitable for a consistent description of the flow, the main part of luminosity can still be carried by matter.

26.2. Collimation

The magnetic field in the jet is globally expansive, corresponding to the fact that it represents a positive energy density. That is, a magnetic field can only exist if there is an external agent to take up the stress it exerts.

Since magnetic jets do not collimate themselves, an external agent has to be involved. A constraint can be derived from the observed opening angle θ_∞. Once the flow speed has a Lorentz factor $\Gamma > 1/\theta_\infty$, the different directions in the flow are out of causal contact, and the opening angle does not change any more (at least not until the jet slows down again, for example by interaction with its environment). Collimation must take place at a distance where the Lorentz factor was still less than $1/\theta_\infty$.

Once on its way with a narrow opening angle, a relativistic jet needs no external forces to keep it collimated. Relativistic kinematics guarantees that it can just continue ballistically, with unconstrained sideways expansion. This can be seen in a number of different ways. One of them is the causality argument above, alternatively with a Lorentz transformation. In a frame comoving with the jet the sideways expansion is limited by the maximum sound speed of a relativistic plasma, $c_{\rm s} = c/\sqrt{3}$. Since transversal to the flow, the apparent expansion rate in a lab frame (a frame co-moving with the central engine) is reduced by a factor Γ: the time dilatation effect. In the comoving frame, the same effect appears as Lorentz contraction: the jet expands as quickly as it can, but distances to points long its path are reduced by a factor ęč (for example the distance to the lobes: the place where jet is stopped by the interstellar medium). In AGNs, with Lorentz factors of order ~ 20, the jet cannot expand to an angle of more than about a degree. This holds if the flow was initially collimated: it still requires that a sufficiently effective collimating agent is present in the region where the jet is accelerated. Comparisons of Lorentz factors and opening angles of AGN jets might provide possible clues on this agent.

The agent responsible for collimation somehow must be connected with the accretion disk (especially in microquasars where there is essentially nothing else around). One possibility is that the observed jet is confined by a slower outflow from the accretion disk.

Another possibility is that the collimation is due to an ordered magnetic field kept in place by the disk: the field that launches the jet from the center may may be part of a larger field configuration that extends, with declining strength, to larger distances in disk (see Fig. 40). If the strength of this field scales with the gas pressure in the disk, one finds that the field lines above the disk naturally have a nearly perfectly collimating shape. The presence and absence of well-defined jets at certain X-ray states would then be related to the details of how ordered magnetic fields are accreted through the disk. Near the compact object, the accretion can be in the form of an ion-supported flow (with ion temperatures near virial) which is geometrically thick ($H/R \sim 1$). Jets launched in the central funnel of such a disk are confined by the surrounding thick accretion flow. As shown by current numerical simulations, this can lead to a fair degree of collimation, though collimation to angles of a few degrees and less as observed in some sources will probably require an additional mechanism. In addition, inverse Compton losses limit the maximum Lorentz factor to less than 5.

26.3. Stability

Jets can be unstable under different conditions. In the collimation phase, the jet has already a B_ϕ component that cannot be neglected dynamically. This configuration is unstable to the kink-mode, among the different modes shown by the instability affecting a magnetized (cylindrical) jet. The instability propagates through the plasma at $\sim v_{\rm A}$. If $c_{\rm s} < v_{\rm A}$, it can destroy the jet. Actually, jets that are conical or with slow expansion will suffer eventually disruption due to the kink-mode of the MHD instability. Jets that expand faster than conical, or with very low magnetic fields (i.e. $v_{\rm A} < c_{\rm s}$), will be stable. The kink stability is more likely to occur relatively close to the black hole, at $\lesssim 100 - 1000\ R_{\rm sch}$, since there B_ϕ is expected to be larger. At larger distances, jet acceleration (e.g. Komissarov et al. 2007) will reduce B_ϕ down to values below equipartition with matter pressure (i.e. $v_{\rm A} < c_{\rm s}$).

There are other ways to disrupt the jet. At some distance the external pressure may become significant compared to the jet lateral ram (thermal+kinetic) pressure. Note that outside the accretion disk atmosphere or the corona, the jet is likely overdense and in overpressure compared with the external medium, but expands and its total pressure decreases. Therefore, at the stage when the jet thermal pressure becomes equal to the external pressure, a recollimation shock generates. If this shock repeats few times or is strong enough, something that de-

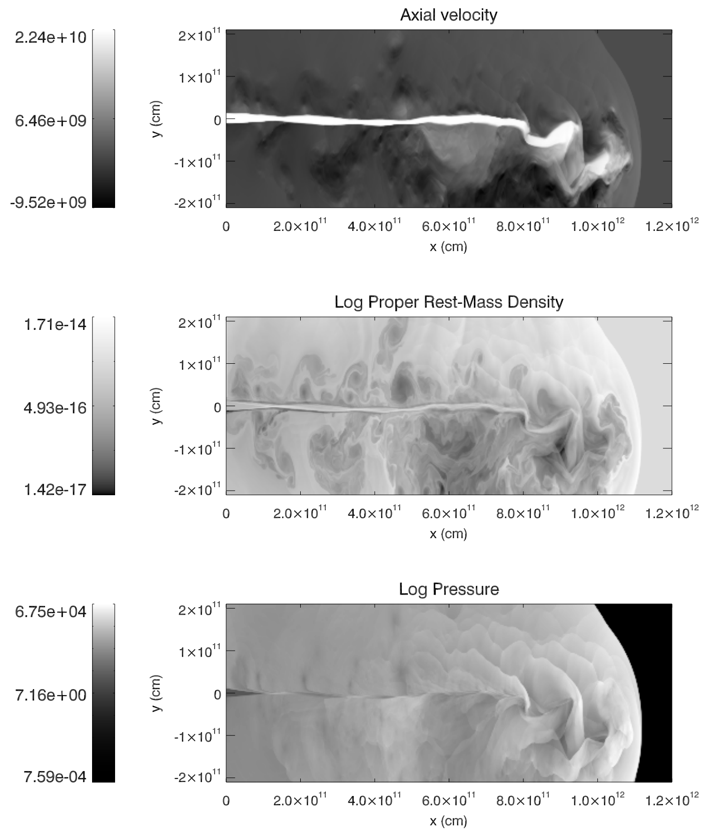

Figure 41. Propagation of a jet through the stellar wind in a massive X-ray binary. Simulations by Perucho and Bosch-Ramon (2008).

pends on the initial pressure ratio, can lead to the disruption of the jet (Perucho & Bosch-Ramon 2008). Also, when there is a strong velocity difference between the external medium and the jet, Kelvin-Helmholtz instabilities can develop, distorting and eventually destroying the jet (Romero 1995). Axial magnetic fields can help to stabilize the jets against the perturbations.

26.4. Content and radiation

The jet can be dominated, in the sense of power, by a cold (thermal) $p - e^{-}$-plasma plus a small contribution of relativistic particles mixed with the outflow, or by a pure relativistic $e^{\pm}$-plasma. The former is expected to occur in hydrodynamic (HD) and magneto-hydrodynamic (MHD) jets (such jets are called 'hadronic' or 'heavy' jets), whereas the latter seems more likely to happen in a electromagnetic (EM) jets (pure 'leptonic' jets). Accretion loaded or black-hole rotation powered jets with medium entrainment, can produce hadronic jets, whereas black-hole rotation powered jets with a diluted medium could lead to leptonic jets. These leptonic jets would consist basically of a powerful (collimated) electromagnetic wave carrying just some $e^{\pm}$ injected by pair creation at the base of the jet.

In HD and low magnetization MHD jets, the particle kinetic energy associated with the motion in the jet direction (i.e. the bulk motion; $e_{\rm b} = (\Gamma_{\rm j} - 1)\, m\, c^2$) is expected to be much larger than the energy associated with jet expansion (i.e. $e_{\rm exp} \approx m\, v_{\rm exp}^2/2$) and temperature (i.e. $e_{\rm kT} = k\, T$). This implies strong collimation, and a low sound ($c_{\rm s}$) or Alfven speed ($v_{\rm A}$) in the jet plasma compared to the jet velocity, i.e. the jet will be strongly supersonic/superalfvenic. If this were not the case, the jet could still be collimated by external pressure, but would be difficult to keep it collimated on long distances. For jets with dynamics dominated by the EM fields the temperature of the plasma could be very large as long as collimation is satisfied, i.e. the external and magnetic pressure should be well above the thermal pressure. Otherwise, the flow would be uncollimated. The realization of the different situations may take place in each jet at different stages of their evolution. The jet could be EM/leptonic at its very base, MHD/hadronic through external medium entrainment at intermediate scales, and HD at the largest scales after magnetic energy has gone to accelerate the jet.

As long as jets are radiatively efficient, they will shine across the EM spectrum and could be detected, but when the conditions in the plasma are not suitable for the production of significant radiation, they may keep dark all the way to their termination regions. The radiation from jets can be thermal (continuum and lines), although the detection of thermal jets is rare since the required densities are high, and T and $v_{\rm j}$ moderate. The detection of non-thermal radiation from jets, mainly in the radio band, is more common. For that, it is required that some particles will be accelerated up to relativistic energies, well above $k\, T$, and at least the presence of magnetic field, since these are the ingredients of synchrotron emission. Efficient high-energy and very high-energy emission is possible if the radiation and/or the matter fields are dense enough. This radiation can be produced by inverse Compton scattering, when accelerated leptons are present, or by proton-proton collisions, when accelerated hadrons are present (e.g. Bosch-Ramon et al. 2006; Romero et al. 2003). For a comprehensive description, including particle transport, see Vila & Romero (2010) and Reynoso et al. (2010).

27. Evidence for black holes in the universe

28. White holes and gravastars

The analytical extension of the Schwarzschild solution in Kruskal-Szekeres coordinates shows a region that is singular and covered by an horizon but from where geodesic lines emerge. Such an extension can be interpreted as a time-reversal image of a black hole: the matter from an expanding cloud began to expand *from* the horizon. Such space-time regions are known as *white holes* (Novikov 1964, Ne'eman 1965). White holes cannot result from the collapse of physical objects in the real Universe, but they could be imagined to be intrinsic features of space-time. A white hole acts as a source that ejects matter from its event horizon. The sign of the acceleration is invariant under time reversal, so both black and

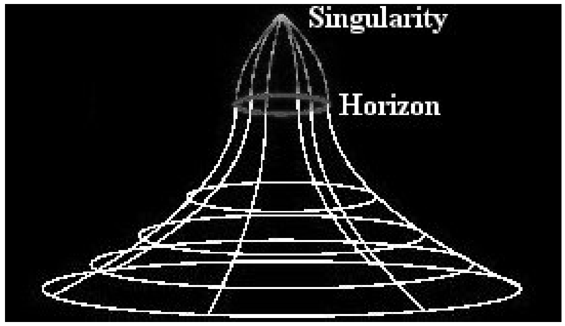

Figure 42. Embedding diagram representing a white hole.

white holes attract matter. The only potential difference between them is in the behavior at the horizon. White hole horizons recede from any incoming matter at the local speed of light, in such a way that the infalling matter never crosses the horizon. The infalling matter is then scattered and re-emitted at the death of the white hole, receding to infinity after having come close to the final singular point where the white hole is destroyed. The total proper time until an infalling object encounters the singular endpoint is the same as the proper time to be swallowed by a black hole, so the white hole picture does not say what happens to the infalling matter. In Figure 42 we show an embedding diagram of a white hole.

The existence of white holes is doubtful since they are unstable. The instability of white holes results from both classical processes caused by the interaction of the surrounding matter (Frolov 1974) and from processes of quantum particle creation in the gravitational field of the holes (Zeldovich et al. 1974). The accretion of matter into white holes causes the instability and converts them into black holes. The reader is referred to Frolov & Novikov (1998) for a detailed discussion. In addition, we notice that the entropy of a black hole is related to the horizon area in Planck units, and this is the most entropy which a given region can contain. When an object flies out of a white hole, the area of the horizon always decreases by more than the maximum possible entropy that can be squeezed into the object, which is a time-reversed statement of the Bekenstein bound[17]. White holes, then, appear to violate the second law of thermodynamics.

A different type of hypothetical objects are Gravitational Vacuum Stars or "gravastars". Gravastars were first proposed by Mazur & Mottola (2001) as an alternative to black holes. They are mathematically constructed as compact objects with an interior de Sitter condensate phase and an exterior Schwarzschild geometry of arbitrary total mass M. These are separated by a phase boundary with a small but finite thickness of fluid with equation of state $p = +\rho c^2$, replacing both the Schwarzschild and de Sitter classical horizons. The interior region has an equation of state $p = -\rho c^2$, in such a way that the energy conditions are

[17]The Bekenstein bound says that the maximum possible entropy of a black hole is $S = A/4$, where A is the two-dimensional area of the black hole's event horizon in units of the Planck area.

violated and the singularity theorems do not apply. The solution, then, has no singularities and no event horizons. The system is sustained against collapse by the negative pressure of the vacuum. The assumption required for this solution to exist is that gravity undergoes a vacuum rearrangement phase transition in the vicinity of $r = r_{\rm Schw}$. Recent theoretical work, however, has shown that gravastars as well as other alternative black hole models are not stable when they rotate (Cardoso et al. 2008). This can be interpreted as a "no go theorem" for them.

29. Wormholes

A wormhole is a region of space-time with non-trivial topology. It has two mouths connected by a throat (see Figure 43). The mouths are not hidden by event horizons, as in the case of black holes, and, in addition, there is no singularity to avoid the passage of particles, or travelers, from one side to the other. Contrary to black holes, wormholes are holes in space-time, i.e. their existence implies a multiple-connected space-time.

There are many types of wormhole solutions for the Einstein field equations (see Visser 1996). Let us consider the static spherically symmetric line element,

$$ds^2 = e^{2\Phi(l)}c^2dt^2 - dl^2 - r(l)^2 d\Omega^2$$

where l is a proper radial distance that covers the entire range $(-\infty, \infty)$. In order to have a wormhole which is traversable in principle, we need to demand that:

1. $\Phi(l)$ be finite everywhere, to be consistent with the absence of event horizons.

2. In order for the spatial geometry to tend to an appropriate asymptotically flat limit, it must happen that

$$\lim_{r\to\infty} r(l)/l = 1$$

and

$$\lim_{r\to\infty} \Phi(l) = \Phi_0 < \infty.$$

The radius of the wormhole is defined by $r_0 = \min\{r(l)\}$, where we can set $l = 0$.

To consider wormholes which can be traversable in practice, we should introduce additional engineering constraints. Notice that for simplicity we have considered both asymptotic regions as interchangeable. This is the best choice of coordinates for the study of wormhole geometries because calculations result considerably simplified. In general, two patches are needed to cover the whole range of l, but this is not noticed if both asymptotic regions are assumed similar. The static line element is:

$$ds^2 = e^{2\Phi(r)}c^2dt^2 - e^{2\Lambda(r)}dr^2 - r^2 d\Omega^2, \tag{363}$$

where the redshift function Φ and the shape-like function $e^{2\Lambda}$ characterize the wormhole topology. They must satisfy:

1. $e^{2\Lambda} \geq 0$ throughout the space-time. This is required to ensure the finiteness of the proper radial distance defined by $dl = \pm\, e^{\Lambda}\, dr$. The $\pm$ signs refer to the two asymptotically flat regions which are connected by the wormhole throat.

2. The precise definition of the wormhole's throat (minimum radius, $r_{\rm th}$) entails a vertical slope of the embedding surface

$$\lim_{r \to r_{\rm th}^+} \frac{dz}{dr} = \lim_{r \to r_{\rm th}^+} \pm \sqrt{e^{2\Lambda} - 1} = \infty. \qquad (364)$$

3. As $l \to \pm\infty$ (or equivalently, $r \to \infty$), $e^{2\Lambda} \to 1$ and $e^{2\Phi} \to 1$. This is the asymptotic flatness condition on the wormhole space-time.

4. $\Phi(r)$ needs to be finite throughout the space-time to ensure the absence of event horizons and singularities.

5. Finally, the *flaring out* condition, that asserts that the inverse of the embedding function $r(z)$ must satisfy $d^2r/dz^2 > 0$ at or near the throat. Stated mathematically,

$$-\frac{\Lambda'\, e^{-2\Lambda}}{(1 - e^{-2\Lambda})^2} > 0. \qquad (365)$$

   This is equivalent to state that $r(l)$ has a minimum.

Static wormhole structures as those described by the above metric require that the average null energy condition must be violated in the wormhole throat. From the metric coefficients can be established (e.g. Morris and Thorne 1988, Visser 1996):

$$G_{tt} + G_{rr} < 0, \qquad (366)$$

where G_{tt} and G_{rr} are the time and radial components of the Einstein tensor: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$.

This constraint can be cast in terms of the stress-energy tensor of the matter threading the wormhole. Using the field equations, it reads

$$T_{tt} + T_{rr} < 0, \qquad (367)$$

which represents a violation of the null energy condition. This implies also a violation of the weak energy condition (see Visser 1996 for details). Plainly stated, it means that the matter threading the wormhole must exert gravitational repulsion in order to stay stable against collapse. Although there are known violations to the energy conditions (e.g. the Casimir effect), it is far from clear at present whether large macroscopic amounts of "exotic matter" exist in nature. If natural wormholes exist in the universe (e.g. if the original topology after the Big-Bang was multiply connected), then there should be observable electromagnetic signatures (e.g. Torres et al. 1998). Currently, the observational data allow to establish an upper bound on the total amount of exotic matter under the

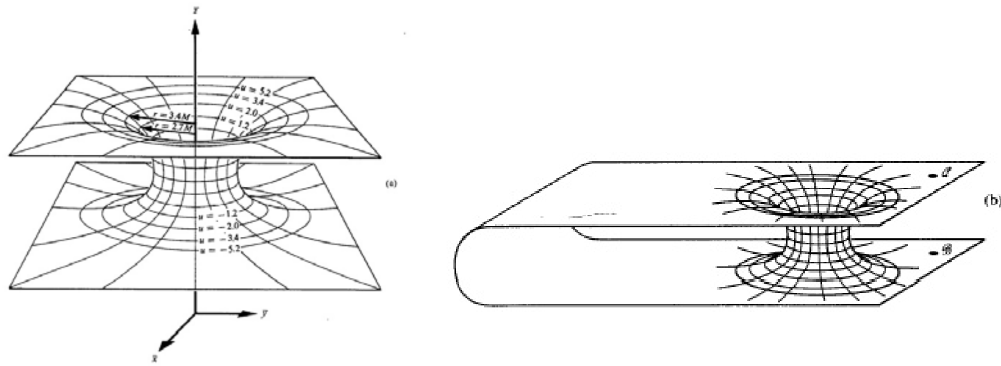

Figure 43. Embedding diagrams of wormholes. Adapted from Misner et al. (1973).

form of wormholes of $\sim 10^{-36}$ g cm^{-3}. The production of this kind of matter in the laboratory is completely out of the current technical possibilities, at least in significant macroscopic quantities.

A simple selection of $\Phi(r)$ and $\Lambda(r)$ are (e.g. Morris and Thorne 1988, Hong and Kim 2006) is:

$$\Phi(r) = \frac{1}{2} \ln \left(1 - \frac{b(r)}{r}\right), \tag{368}$$

$$e^{2\Lambda(r)} = \left(1 - \frac{b(r)}{r}\right)^{-1}, \tag{369}$$

where

$$b(r) = b(r_0) = \text{const} = B > 0. \tag{370}$$

The wormhole shape function has a minimum at $r = r_0$ where the exotic matter is concentrated.

Another possibility is the so-called "absurdly benign" wormhole (Morris and Thorne 1988):

$$b(r) = b_0 \left[\frac{1 - (r - b_0)}{a_0}\right]^2, \quad \Phi(r) = 0, \quad \text{for} \quad b_0 \leq r \leq b_0 + a_0, \tag{371}$$

$$b = \Phi = 0, \quad \text{for} \quad r \geq b_0 + a_0. \tag{372}$$

Finally, we mention that a wormhole can be immediately transformed into a time machine inducing a time-shift between the two mouths. This can be made through relativistic motion of the mouths (a special relativity effect) or by exposing one of them to an intense gravitational field (see Morris and Thorne 1988 and Morris et al. 1988 for further details; for the paradoxes of time travel, see Romero and Torres 2001).

30. Closed time-like curves and time travel

Closed time-like curves (CTCs) are worldlines of any physical system in a temporally orientable space-time which, moving always in the future direction, ends

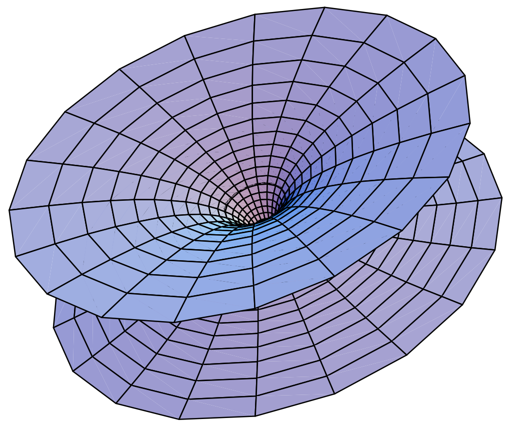

Figure 44. Embedding of a Lorentzian wormhole.

arriving back at some point of its own past. Although solutions of the Einstein field equations where CTCs exist are known at least since Gödel's (1949) original work on rotating universes, it has been only since the last decade of the XX Century that physicists have shown a strong and sustained interest on this topic. The renewed attraction of CTCs and their physical implications stem from the discovery, at the end of the 1980's, of traversable wormhole space-times (Morris et al. 1988).

Any space-time $(M,\ g_{ab})$ with CTCs is called a chronology-violating space-time. There are two types of these space-times: those where CTCs exist everywhere (like, for instance, in Gödel space-time), and those where CTCs are confined within some regions and there exists at least one region free of them. The regions with CTCs are separated from the "well-behaved" space-time by Cauchy horizons (wormhole space-times belong to this latter type). Here we shall restrict the discussion to the second type of space-times

The existence of CTCs and the possibility of backward time travel have been objected by several scientists championed by Hawking (1992), who proposed the so-called chronology protection conjecture: the laws of physics are such that the appearance of CTCs is never possible. The suggested mechanism to enforce chronology protection is the back-reaction of vacuum polarization fluctuations: when the renormalized energy-momentum tensor is fed back to the semi-classical Einstein field equations, the back-reaction accumulates energy in such a way that it may distort the space-time geometry so strongly as to form a singularity, destroying the CTC at the very moment of its formation.

It has been argued, however, that quantum gravitational effects would cut the divergence off saving the CTCs (Kim and Thorne 1991). By other hand, Li et al. (1993) pointed out that the divergence of the energy-momentum tensor does not prevent the formation of a CTC but is just a symptom that a full quantum gravity theory must be applied: singularities, far from being physical entities that can act upon surrounding objects, are manifestations of the breakdown of the gravitational theory. In any case, we cannot draw definitive conclusions with the semi-classical tools at our disposal (see Earman 1995 for additional discussion).

But even if the energy-momentum tensor of vacuum polarization diverges at the Cauchy horizon it is not necessarily implied that CTCs must be destroyed, since the equations can be well-behaved in the region inside the horizon (Li et al. 1993). In particular, wormhole space-times could be stabilized against vacuum fluctuations introducing reflecting boundaries between the wormhole mouths (Li 1994) or using several wormholes to create CTCs (Thorne 1992, Visser 1997).

Even Hawking has finally recognized that back-reaction does not necessarily enforce chronology protection (Cassidy and Hawking 1998). Although the quest for finding an effective mechanism to avoid CTCs continues, it is probable that the definitive solution to the problem should wait until a complete theory of quantum gravity can be formulated. In the meantime, the profound physical consequences of time travel in General Relativity should be explored in order to push this theory to its ultimate limits, to the region where the very foundations of the theory must be revisited.

A different kind of objection to CTC formation is that they allow illogical situations like the "grandfather" paradox[18] which would be expressing that the corresponding solutions of the field equations are "non-physical". This is a common place and has been conveniently refuted by Earman (1995), among others (see also Nahin 1999 and references therein). Grandfather-like paradoxes do not imply illogical situations. In particular, they do not mean that local determinism does not operate in chronology-violating space-times because it is always possible to choose a neighborhood of any point of the manifold such that the equations that represent the laws of physics have appropriate solutions. Past cannot be changed (the space-time manifold is unique) but it can be causally affected from the future, according to General Relativity. The grandfather paradox, as pointed out by Earman 1995, is just a manifestation of the fact consistency constraints must exist between the local and the global order of affairs in space-time. This leads directly to the so-called Principle of Self-Consistency (PSC).

30.1. The Principle of Self-Consistency

In space-times with CTCs, past and future are no longer globally distinct. Events on CTCs should causally influence each other along a time-loop in a self-adjusted, consistent way in order to occur in the real universe. This has been stated by Friedman et al. (1990) as a general principle of physics:

Principle of Self-Consistency: The only solutions to the laws of physics that occur locally in the real universe are those which are globally self-consistent.

When applied to the grandfather paradox, the PSC says that the grandfather cannot be killed (a local action) because in the far future this would generate an inconsistency with the global world line of the time traveler. Just consistent histories can develop in the universe. An alternative way to formulate the PSC

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[18]The grandfather paradox: A time traveler goes to his past and kills his young grandfather then avoiding his own birth and, consequently, the time travel in which he killed his grandfather.

is to state that (Earman 1995):

The laws of physics are such that any local solution of their equations that represents a feature of the real universe must be extensible to a global solution.

The principle is not tautological or merely prescriptive, since it is clear that local observations can provide information of the global structure of the world: it is stated that there is a global-to-local order in the universe in such a way that certain local actions are ruled out by the global properties of the space-time manifold.

If the PSC is neither a tautological statement nor a methodological rule, what is then its epistemological status? It has been suggested that it could be a basic law of physics –in the same sense that the Einstein field equations are laws of physics– (Earman 1995). This would imply that there is some "new physics" behind the PSC. By other hand, Carlini et al. (1995) have proposed, on the basis of some simple examples, that the PSC could be a consequence of the Principle of Minimal Action. In this case, no new physics would be involved. Contrary to these opinions, that see in the PSC a law statement, or at least a consequence of law statements, we suggest that this principle actually is a *metanomological statement*, like the Principle of General Covariance among others (see Bunge 1961 for a detailed discussion of metanomological statements). This means that the reference class of the PSC is not formed by physical systems, but by laws of physical systems. The usual laws are restrictions to the state space of physical systems. Metanomological statements are laws of laws, i.e. restrictions on the global network of laws that thread the universe. The requirement of consistency constraints would then be pointing out the existence of deeper level super-laws, which enforce the harmony between local and global affairs in space-time. Just in this sense it is fair to say that "new physics" is implied.

30.2. Causal loops: Self-existent objects

Although the PSC eliminates grandfather-like paradoxes from chronology-violating space-times, other highly perplexing situations remain. The most obscure of these situations is the possibility of an ontology with self-existent objects. Let us illustrate with an example what we understand by such an object:

Suppose that, in a space-time where CTCs exist, a time traveler takes a ride on a time machine carrying a book with her. She goes back to the past, forgets the book in -what will be- her laboratory, and returns to the future. The book remains then hidden until the time traveler finds it just before starting her time trip, carrying the book with her.

It is not hard to see that the primordial origin of the book remains a mystery. Where does the book come from?. This puzzle has been previously mentioned in philosophical literature by Nerlich (1981) and MacBeath (1982). Physicists, instead, have not paid much attention to it, despite the interesting fact that the described situation is apparently not excluded by the PSC: the local and global structures of the loop are perfectly harmonious and there is no causality violations. There is just a book never created, never printed, but, somehow, existing in space-time. It has been suggested (Nerlich 1981) that if CTCs exist, then we are committed to accept an ontology of self-existing objects: they are just out there, trapped in space-time. There is no sense in asking where they are from. Even energy is conserved if we admit that the system to be considered is not only the present time-slice of the manifold but rather the two slices connected by the time loop: the energy removed from the present time ($M_{\mathrm{book}}c^2$) is deposited in the past.

However, that the acceptance of such a bizarre ontology proceeds from an incorrect application of the PSC. This principle is always discussed within the context of General Relativity, although actually it should encompass *all* physical laws. A fully correct formulation of the PSC should read *laws of nature* where in the formulation given above it is said *laws of physics*. What should be demanded is total consistency and not only consistency in the solutions of the Einstein field equations. In particular, when thermodynamics is included in the analysis, the loop of a self-existing objects becomes inconsistent because, due to entropic degradation, the final and initial states of the object do not match (Romero & Torres 2001). Moreover, even more strange paradoxes, related to human self-reproduction like the amazing Jocasta paradox (Harrison 1979), can be proven to be non-consistent when the laws of genetics are taken into account (see Nahin 1999 and references therein).

30.3. Information loops and the PSC

Consider the local light cone of a time traveler. There are three, and only three, possible final destinations for a backwards time trip. The arrival point could be **a** within the past light cone, **b** on the edge of it, or **c** elsewhere out of the cone. In the case **a** the time traveler can transmit information at a velocity $v \leq c$ and affect its own past. Information transmitted in case **b** that propagates at the velocity of light, instead, will arrive at the very moment when the time travel started. In case **c**, the information flux can only reach the future of the time traveler. However, even in the last case, if more than one time machine are available (for instance as in the situation known as a Roman ring, where there are two wormholes in relative motion) the past might be affected by information flux from the future. The conclusion, then, is that whatever the final destiny of the time-traveler is, in principle, she could always affect her own past. Otherwise stated, chronology-violating space-times generally admit information loops: they cannot be excluded on the only basis of the PSC.

Although the PSC does not preclude that within chronology-violating regions information steaming from the future can affect the past, it at least imposes constraints on the way this can be done. In fact, any physical process causally triggered by the backwards information flux must be consistent with the past history of the universe. This means that if a time traveler goes back to his past and tries, for instance, to communicate the contents of the theory of special relativity to the scientific community before 1905, she will fail *because* at her departure it was historically clear that the first paper on special relativity was published by Albert Einstein in June 1905. The details of her failure will depend on the details of her travel and attempt, in the same way that the details of the failure of a perpetual motion machine depends on the approach used by the imprudent inventor. All we can say *a priori* is that the laws of physics are such that these attempts *cannot* succeed and information cannot propagate arbitrarily in space-time. It is precisely due to the PSC that we know that our past is not significantly affected from the future: we know that till now, knowledge has been generated by evolutionary processes, i.e. there is small room in history for information loops. This does not necessarily mean that the same is valid for the entire space-time.

31. The future of black holes

According to Eq. (255), an isolated black hole with $M = 10\ M_{\odot}$ would have a lifetime of more than 10^{66} yr. This is 56 orders of magnitude longer than the age of the universe[19]. However, if the mass of the black hole is small, then it could evaporate within the Hubble time. A primordial black hole, created by extremely energetic collisions short after the Big Bang, should have a mass of at least 10^{15} g in order to exist today. Less massive black holes must have already evaporated. What happens when a black hole losses its mass so it cannot sustain an event horizon anymore?. As the black hole evaporates, its temperature raises. When it is cold, it radiates low energy photons. When the temperature increases, more and more energetic particles will be emitted. At some point gamma rays would be produced. If there is a population of primordial black holes, their radiation should contribute to the diffuse gamma-ray background. This background seems to be dominated by the contribution of unresolved Active Galactic Nuclei and current observations indicate that if there were primordial black holes their mass density should be less than $10^{-8}\ \Omega$, where Ω is the cosmological density parameter (~ 1). After producing gamma rays, the mini black hole would produce leptons, quarks, and super-symmetric particles, if they exist. At the end the black hole would have a quantum size and the final remnant will depend on the details of how gravity behaves at Planck scales. The final product might be a stable, microscopic object with a mass close to the Planck mass. Such particles might contribute to the dark matter present in the Galaxy and in other galaxies and clusters. The cross-section of black hole relics is extremely small: 10^{-66} cm^2 (Frolov and Novikov 1998), hence they would be basically non-interacting particles.

A different possibility, advocated by Hawking (1976), is that, as a result of the evaporation nothing is left behind: all the energy is radiated. This creates a puzzle about the fate of the information stored in the black hole: is it radiated away during the black hole lifetime or does it simply disappear from the universe?.

Actually, the very question is likely meaningless: information is not a property of physical systems. Information is a property of languages, and languages are human constructs. The physical property usually confused with information is entropy. The reason is that a same mathematical formalism can be used to describe both properties. Of course, this does not mean that these quite different properties are identical. Black holes, as we have seen, have huge entropy. Is

[19] We assume that the universe originated at the Big Bang, although, of course, this needs not to be necessarily the case.

entropy of the universe decreasing when a black hole evaporates?. I think that the answer is the same one given by Bekenstein and already mentioned: *the total generalized entropy never decreases*. The entropy of the universe was increasing due to the black hole evaporation through a simple process of thermalization. The disappearance of the horizon is simply the end of such a process. Information is related to our capability of describing the process through a mathematical language, not to the process itself.

Independently of the problem of mini black hole relics, it is clear that the fate of stellar-mass and supermassive black holes is related to fate of the whole universe. In an ever expanding universe or in an accelerating universe as it seems to be our actual universe, the fate of the black holes will depend on the acceleration rate. The local physics of the black hole is related to the cosmic expansion through the cosmological scale factor $a(t)$, which is an arbitrary (positive) function of the co-moving time t. A Schwarzschild black hole embedded in a Friedmann-Robertson-Walker universe can be represented by a generalization of the McVittie metric (e.g. Gao et al. 2008):

$$ds^2 = \frac{\left[1 - \frac{2GM(t)}{a(t)c^2 r}\right]^2}{\left[1 + \frac{2GM(t)}{a(t)c^2 r}\right]^2} c^2 dt^2 - a(t)^2 \left[1 + \frac{2GM(t)}{a(t)c^2 r}\right]^4 (dr^2 + r^2 d\Omega^2). \tag{373}$$

Assuming that $M(t) = M_0 a(t)$, with M_0 a constant, the above metric can be used to study the evolution of the black hole as the universe expands. It is usual to adopt an equation of state for the cosmic fluid given by $P = \omega \rho c^2$, with ω constant. For $\omega < -1$ the universe accelerates its expansion in such a way that the scale factor diverges in a finite time. This time is known as the Big Rip. If $\omega = -1.5$, then the Big Rip will occur in 35 Gyr. The event horizon of the black hole and the cosmic apparent horizon will coincide for some time $t < t_{\rm Rip}$ and then the singularity inside the black hole would be visible to observers in the universe. Unfortunately for curious observers, Schwarzschild black holes surely do not exist in nature, since all astrophysical bodies have some angular momentum and is reasonable then to expect only Kerr black holes to exist in the universe. Equation (373) does not describe a cosmological embedded Kerr black hole. Although no detailed calculations exist for such a case, we can speculate that the observer would be allowed to have a look at the second horizon of the Kerr black hole before being ripped apart along with the rest of the cosmos. A rather dark view for the Doomsday.

32. Conclusions

Altogether, it is surely *darker than you think*[20].

Acknowledgments. I would like to thank V. Bosch-Ramon, Ernesto Eiroa, Daniela Pérez, Santiago E. Perez-Bergliaffa, and Brian Punsly for several discussions on black holes and related topics.

[20] Jack Williamson, Fantasy Press, 1948.

Appendix A.1. Manifolds and topology

Appendix A.2. General definition of space-time

Appendix A.3. The anisotropy of time

The electromagnetic radiation can be described in the terms of the 4-potential A^{μ}, which in the Lorentz gauge satisfies:

$$\partial^{\nu}\partial_{\nu}A^{\mu}(\vec{r},\ t) = 4\pi j^{\mu}(\vec{r},\ t), \tag{374}$$

with $c = 1$ and j^{μ} the 4-current. With appropriate boundary conditions is possible to write A^{μ} as a functional of the sources j^{μ}. The retarded and advanced solutions are:

$$A^{\mu}_{\rm ret}(\vec{r},\ t) = \int \frac{j^{\mu}\left(\vec{r},\ t - \left|\vec{r} - \vec{r'}\right|\right)}{\left|\vec{r} - \vec{r'}\right|} d^3\vec{r'}, \tag{375}$$

$$A^{\mu}_{\rm adv}(\vec{r},\ t) = \int \frac{j^{\mu}\left(\vec{r},\ t + \left|\vec{r} - \vec{r'}\right|\right)}{\left|\vec{r} - \vec{r'}\right|} d^3\vec{r'}. \tag{376}$$

The two functionals of $j^{\mu}(\vec{r},\ t)$ are related to one another by a time reversal transformation. The solution (375) is contributed by all sources in the causal past of the space-time point $(\vec{r},\ t)$ and the solution (375) by all the sources in the causal future of that point. The linear combinations of these solutions are also solutions, since the equations are linear and the Principle of Superposition holds. It is usual to consider only the retarded potential as physical meaningful in order to estimate the electromagnetic field at $(\vec{r},\ t)$: $F^{\mu\nu}_{\rm ret} = \partial^{\mu}A^{\nu}_{\rm ret} - \partial^{\nu}A^{\mu}_{\rm ret}$. However, there seems to be no compelling reason for such a choice. We can adopt, for instance, (in what follows we use a simplified notation and boundary conditions such as the surface contribution is zero):

$$A^{\mu}(\vec{r},\ t) = \frac{1}{2}\int_V (\text{adv}\ +\ \text{ret})\ dV. \tag{377}$$

If the space-time is curve ($R \neq 0$), the null cones that determines the local causal structure will not be symmetric around the point $p\ (\vec{r},\ t)$. Then,

$$L^{\mu} = \lim_{V\to\infty}\left[\int_V \text{ret} - \int_{\rm V} \text{adv}\right]\ dV \neq 0. \tag{378}$$

If $g_{\mu\nu}L^{\mu}T^{\nu} \neq 0$, with $T^{\nu} = (1, 0, 0, 0)$ there is a preferred direction for flow of the Poynting flux in space-time. In a black hole interior this direction is towards the singularity. In an expanding Universe, it is in the global future direction. We see, then, that time, in a general space-time (M, g_{ab}), is *anisotropic*. There is a global to local relation given by the Poynting flux as determined by the curvature of space-time that indicates the direction in which events occur. Physical processes, inside a black hole, have a different orientation that outside, and the causal structure of the world is determined by the dynamics of space-time and the initial conditions. Macroscopic irreversibility[21] and time anisotropy emerge from essentially reversible laws. For additional details see Romero (2010).

Appendix A.4. The ontology of the world

The ontology is the part of metaphysics that deals with the most general features of the world. It essentially provides theories about the structure and basic contents of the world. These theories are presupposed by physical theories, which are concerned with the detailed laws and specific characteristics of whatever exists.

The concept of individual is the basic primitive concept of any ontological theory. Individuals associate themselves with other individuals to yield new individuals. It follows that they satisfy a calculus, and that they are rigorously characterized only through the laws of such calculus. These laws are set with the aim of reproducing the way real things associate. Specifically, it is postulated that every individual is an element of a set S in such a way that the structure $\mathcal{S} =< S, \circ, \Box >$ is a *commutative monoid of idempotents*. In the structure $\mathcal{S}$, S is to be interpreted as the set of all the individuals, the element $\Box \in S$ as the null individual, and the binary operation $\circ$ as the association of individuals. It is easy to see that there are two classes of individuals: *simple* and *composed*.

D$_1$ $x \in S$ is composed $\Leftrightarrow \exists\, y, z \in S / x = y \circ z$.

D$_2$ $x \in S$ is simple $\Leftrightarrow^{\neg} \exists\, y, z \in S / x = y \circ z$.

D$_3$ $x \sqsubset y \Leftrightarrow x \circ y = y$ (x is part of $y \Leftrightarrow x \circ y = y$).

D$_4$ $\mathcal{C}(x) \equiv \{y \in S / y \sqsubset x\}$ (composition of x).

Real things differentiate from abstract individuals because they have a number of properties in addition to their capability of association. These properties can be *intrinsic* ($P_{\rm i}$) or *relational* ($P_{\rm r}$). The intrinsic properties are inherent and they are represented by predicates or unary applications, while relational properties are represented by n-ary predicates, with n>1, as long as nonconceptual arguments are considered. For instance, the position and the velocity of a particle are relational properties, but its charge is an intrinsic property.

P is called a *substantial property* if and only if some individual x possesses P:

D$_5$ $P \in \mathcal{P} \Leftrightarrow (\exists x)(x \in S \wedge Px)$.

Here $\mathcal{P}$ is the set of all the substantial properties. The set of the properties of a given individual x is

D$_6$ $P(x) \equiv \{P \in \mathcal{P} / Px\}$.

[21] Notice that the electromagnetic flux is related with the macroscopic concept of temperature through the Stefan-Boltzmann law: $L = A\sigma_{\rm SB}T^4$, where $\sigma_{\rm SB} = 5.670400 \times 10^{-8}{\rm J\,s^{-1}m^{-2}K^{-4}}$ is the StefanŨBoltzmann constant.

If two individuals have exactly the same properties they are the same: $\forall\ x, y \in S$ if $P(x) = P(y) \Rightarrow x \equiv y$. Two individuals are identical if their intrinsic properties are the same: $x \overset{id}{\leftrightarrow} y$ (they can differ only in their relational properties).

A detailed account of the theory of properties is given in Bunge (1977). We only give here two useful definitions:

D₇ P is an inherited property of $x \Leftrightarrow P \in P(x) \wedge (\exists y)(y \in \mathcal{C}(x) \wedge y \neq x \wedge P \in P(y))$.

D₈ P is an emergent property of $x \Leftrightarrow P \in P(x) \wedge (\ (\forall\ y)_{\mathcal{C}(x)}(y \neq x\ ) \Rightarrow P \notin P(y)\ )$.

According to these definitions, mass is an inherited property and viscosity is an emergent property of a classical fluid.

An individual with its properties make up a *thing* X:

D₉ $X \overset{Df}{=} < x, P(x) >$.

The laws of association of things follow from those of the individuals. The association of all things is the *Universe* (σ_U). It should not be confused with the set of all things; this is only an abstract entity and not a thing.

Given a thing $X =< x, P(x) >$, a conceptual object named *model* X_m of the thing X can be constructed by a nonempty set M and a finite sequence $\mathcal{F}$ of mathematical functions over M, each of them formally representing a property of x:

D₁₀ $X_m \overset{Df}{=} < M, \mathcal{F} >$, where $\mathcal{F} =< \mathcal{F}_1, \ldots, \mathcal{F}_n > / \mathcal{F}_i : M \rightarrow V_i,\ 1 \leq i \leq n, V_i$ vector space , $\mathcal{F}_i \hat{=} P_i \in P(x)$.

It is said then that X_m represents X: $X_m \hat{=} X$ (Bunge 1977).

The *state* of the thing X can be characterized as follows:

D₁₁ Let X be a thing with model $X_m =< M, \mathcal{F} >$, such that each component of the function

$$\mathcal{F} =< \mathcal{F}_1, \ldots, \mathcal{F}_n >: M \rightarrow V_1 \times \ldots \times V_n.$$

represents some $P \in P(x)$. Then $\mathcal{F}_i$ $(1 \leq i \leq n)$ is named i-th state function of X, $\mathcal{F}$ is the *total state function* of X, and the value of $\mathcal{F}$ for some $m \in M$, $\mathcal{F}(m)$, represents the *state of* X *at* m *in the representation* X_m.

If all the V_i, $1 \leq i \leq n$, are vector spaces, $\mathcal{F}$ is the state vector of X in the representation X_m, and $V = V_1 \times \ldots \times V_n$ is the state space of X in the representation X_m.

The concept of physical law can be introduced as follows:

D₁₂ Let $X_m =< X, \mathcal{F} >$ be a model for X. Any restriction on the possible values of the components of $\mathcal{F}$ and any relation between two or more of them is a *physical law* if and only if it belongs to a consistent theory of the X and has been satisfactorily confirmed by the experiment.

We say that a thing X *acts* on a thing Y if X modifies the path of Y in its space state ($X \rhd Y$: X acts on Y).

We say that two things X and Y are *connected* if at least one of them acts on the other. We come at last to the definition of *system*:

$\mathbf{D_{13}}$ A system is a thing composed by at least two connected things.

In particular, a physical system is a system ruled by physical laws. A set of things is not a system, because a system is a physical entity and not a set. A system may posses emergent properties with respect to the component subsystems. The composition of the system σ with respect to a class A of things is (at the instant t):

$$C_A(\sigma, t) = \{X \in A/X \sqsubset \sigma\}.$$

$\mathbf{D_{14}}$ $\overline{\sigma}_A(\sigma, t) = \{X \in A/X \notin \mathcal{C}_A(\sigma, t) \wedge (\exists Y)_{\mathcal{C}_A(\sigma,t)} \wedge (X \rhd Y \vee Y \rhd X)\,)\}$ is the A-environment of σ at t.

If $\overline{\sigma}_A(\sigma, t) = \emptyset \Rightarrow \sigma$ is *closed* at the instant t. In any other case we say that it is open.

A specific physical system will be characterized by making explicit its space of physical states. This is done in the axiomatic basis of the physical theory (e.g. Perez-Bergliaffa et al. 1993, 1996).

Appendix A.5. Annotated bibliography on black holes

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