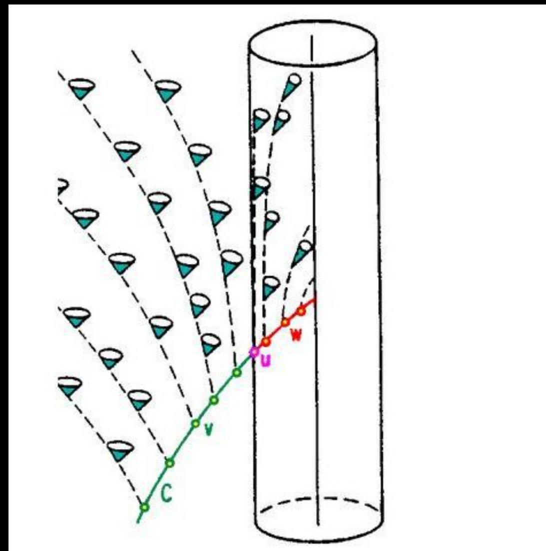

Aplicaciones de GRTensor en Astrofísica y Cosmologia

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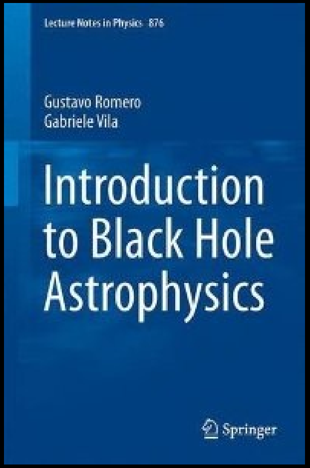
- El agujero negro de Schwarszchild.
- Breve introdución al Maple → movimiento de partículas en la geometria del AN de Schwarszchild.

El agujero negro de Schwarszchild
(solución estática y esfericamente simetrica de las Ecs. de Einstein em ausencia de materia)



Geometria estática com simetria esférica

$$ds^2 = A(r)\,dt^2 - B(r)\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2).$$

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi T^{\mu\nu} \longrightarrow R_{\mu\nu} = 0.$$

$$R_{\mu\nu} = \partial_\nu \Gamma^\sigma{}_{\mu\sigma} - \partial_\sigma \Gamma^\sigma{}_{\mu\nu} + \Gamma^\rho{}_{\mu\sigma}\Gamma^\sigma{}_{\rho\nu} - \Gamma^\rho{}_{\mu\nu}\Gamma^\sigma{}_{\rho\sigma},$$

$$\Gamma^\sigma{}_{\mu\nu} = \tfrac{1}{2}g^{\sigma\rho}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}).$$

$$\Gamma^0{}_{01} = A'/(2A), \qquad \Gamma^1{}_{00} = A'/(2B), \qquad \Gamma^1{}_{11} = B'/(2B),$$

$$\Gamma^1{}_{22} = -r/B, \qquad \Gamma^1{}_{33} = -(r\sin^2\theta)/B, \qquad \Gamma^2{}_{12} = 1/r,$$

$$\Gamma^2{}_{33} = -\sin\theta\cos\theta, \qquad \Gamma^3{}_{13} = 1/r, \qquad \Gamma^3{}_{23} = \cot\theta.$$

$$R_{00} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB},$$

$$R_{11} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB},$$

$$R_{22} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right),$$

$$R_{33} = R_{22}\sin^2\theta.$$

$$ds^2 = c^2\left(1-\frac{2GM}{c^2 r}\right)dt^2 - \left(1-\frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2.$$

Métrica de Schwarzschild

Una forma de entender as características del e-t asociado a una métrica dada es a través de las geodésicas:

$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\sigma}\frac{dx^\rho}{d\sigma} = 0, \longrightarrow x_\mu = x_\mu(\sigma)$$

Las geodésicas extremizan el intervalo entre dos puntos del espacio-tiempo:

$$\tau_{AB} = \int_0^1 d\sigma \left(-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \right)^{1/2} .$$

$$x_\mu = x_\mu(\sigma)$$

$$-\frac{d}{d\sigma}\left(\frac{\partial L}{\partial(dx^\alpha/d\sigma)}\right) + \frac{\partial L}{\partial x^\alpha} = 0,$$

$$L\left(\frac{dx^\alpha}{d\sigma}, x^\alpha\right) = \left(-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \right)^{1/2} .$$

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau}.$$

$$L = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu},$$

$$\dot{x}^{\mu} \equiv dx^{\mu}/d\sigma$$

$$L = c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2\right).$$

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{x}^{\mu}}\right) - \frac{\partial L}{\partial x^{\mu}} = 0.$$

$$\left(1-\frac{2\mu}{r}\right)\dot{t}=k,$$

$$\left(1-\frac{2\mu}{r}\right)^{-1}\ddot{r}+\frac{\mu c^2}{r^2}\dot{t}^2-\left(1-\frac{2\mu}{r}\right)^{-2}\frac{\mu}{r^2}\dot{r}^2-r\left(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2\right)=0,$$

$$\ddot{\theta}+\frac{2}{r}\dot{r}\dot{\theta}-\sin\theta\cos\theta\,\dot{\phi}^2=0,$$

$$r^2\sin^2\theta\,\dot{\phi}=h.$$

(L no depende ni de t ni de ϕ)

$\theta = \pi/2.$

$$\left(1-\frac{2\mu}{r}\right)\dot{t} = k,$$

$$\left(1-\frac{2\mu}{r}\right)^{-1}\ddot{r} + \frac{\mu c^2}{r^2}\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-2}\frac{\mu}{r^2}\dot{r}^2 - r\dot{\phi}^2 = 0,$$

$$r^2\dot{\phi} = h.$$

Interpretación de las constantes

$x \rightarrow x',$ $g'_{\mu\nu}(y) = g_{\mu\nu}(y)$ **Para cualquier *y***

La métrica g' es la misma función de la coordenada x' que la métrica g de x.

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\, g_{\rho\sigma}(x)$$

$$g_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\, g'_{\rho\sigma}(x')$$

$g'_{\rho\sigma}(x') \rightarrow g_{\rho\sigma}(x')$

$$g_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\, g_{\rho\sigma}(x')$$

$x \rightarrow x'$

isometria

$x'^\mu = x^\mu + \varepsilon\xi^\mu(x)$

$|\varepsilon| \ll 1$

$$0 = \frac{\partial \xi^\mu(x)}{\partial x^\rho} g_{\mu\sigma}(x) + \frac{\partial \xi^\nu(x)}{\partial x^\sigma} g_{\rho\nu}(x) + \xi^\mu(x) \frac{\partial g_{\rho\sigma}(x)}{\partial x^\mu}$$

$$\xi_\sigma \equiv g_{\mu\sigma}\xi^\mu$$

$$\begin{aligned} 0 &= \frac{\partial \xi_\sigma}{\partial x^\rho} + \frac{\partial \xi_\rho}{\partial x^\sigma} + \xi^\mu \left[\frac{\partial g_{\rho\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\sigma}}{\partial x^\rho} - \frac{\partial g_{\rho\mu}}{\partial x^\sigma} \right] \\ &= \frac{\partial \xi_\sigma}{\partial x^\rho} + \frac{\partial \xi_\rho}{\partial x^\sigma} - 2\xi_\mu \Gamma^\mu_{\rho\sigma} \end{aligned}$$

$$0 = \xi_{\sigma;\rho} + \xi_{\rho;\sigma}$$

Ecuación de Killing

$$\nabla_{\mathbf{u}} \mathbf{u} = 0,$$

$$\nabla_{\mathbf{u}} (\mathbf{u} \cdot \boldsymbol{\xi}) = u^{\alpha} u^{\mu}{}_{;\alpha} \xi_{\mu} + u^{\alpha} u^{\mu} \xi_{\mu;\alpha}.$$

sim. x antisim.

$$\nabla_{\mathbf{u}} (\mathbf{u} \cdot \boldsymbol{\xi}) = 0.$$ → $\mathbf{p} \cdot \boldsymbol{\xi}$ es constante sobre la geodésica

Si la métrica no depende de la coordenada (cíclica) x^k, el vector de Killing asociado es

$$\xi^{\mu} = \delta^{\mu}{}_{K}.$$

$$\xi_{\mu;\nu} = g_{\mu\alpha}\xi^{\alpha}{}_{;\nu} = g_{\mu\alpha}\left(\frac{\partial \xi^{\alpha}}{\partial x^{\nu}} + \Gamma^{\alpha}_{\nu\sigma}\xi^{\sigma}\right)$$

$$= g_{\mu\alpha}\Gamma^{\alpha}_{\nu K} = \Gamma_{\mu\nu K} = \frac{1}{2}\left(\frac{\partial g_{\mu K}}{\partial x^{\nu}} + \frac{\partial g_{\mu\nu}}{\partial x^{K}} - \frac{\partial g_{\nu K}}{\partial x^{\mu}}\right)$$

$$= \frac{1}{2}(g_{\mu K,\nu} - g_{\nu K,\mu}).$$

$$\mathbf{p}\cdot\boldsymbol{\xi}_{\alpha} = p_{\mu}\xi^{\mu}{}_{\alpha} = p_{\mu}\delta^{\mu}{}_{\alpha} = p_{\alpha}.$$

El momento canónico asociado a la coordenada cíclica es conservado.

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2.$$

→ p_t e p_{φ} son conservados

Em el caso de particulas com masa no nula, podemos parametrizar la geodésica com el tiempo propio, y

$$\left(1-\frac{2\mu}{r}\right)\dot{t}=k,$$

$$c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2-\left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2-r^2\dot{\phi}^2=c^2,$$

$$r^2\dot{\phi}=h.$$

$$\mu = GM/c^2$$

$$\dot{r}^2+\frac{h^2}{r^2}\left(1-\frac{2GM}{c^2r}\right)-\frac{2GM}{r}=c^2(k^2-1),$$

$$k = E/(m_0c^2)$$

$$h = r^2\dot{\phi}$$

$$\frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \frac{h}{r^2}\frac{dr}{d\phi}.$$

$$\left(\frac{h}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{h^2}{r^2} = c^2(k^2-1) + \frac{2GM}{r} + \frac{2GMh^2}{c^2r^3}.$$

$$u \equiv 1/r$$

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{c^2}{h^2}(k^2-1) + \frac{2GMu}{h^2} + \frac{2GMu^3}{c^2}.$$

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2.$$

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2}.$$

Movimiento radial

ϕ = const. $\rightarrow h = 0$
Partícula com velocidad nula
en infinito $\rightarrow k = 1$

$$\frac{dt}{d\tau} = \left(1 - \frac{2\mu}{r}\right)^{-1},$$

$$\frac{dr}{d\tau} = -\left(\frac{2\mu c^2}{r}\right)^{1/2},$$

$$\tau = \frac{2}{3}\sqrt{\frac{r_0^3}{2\mu c^2}} - \frac{2}{3}\sqrt{\frac{r^3}{2\mu c^2}},$$

$\tau = 0$ at $r = r_0$.

$$\frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt} = -\left(\frac{2\mu c^2}{r}\right)^{1/2}\left(1 - \frac{2\mu}{r}\right).$$

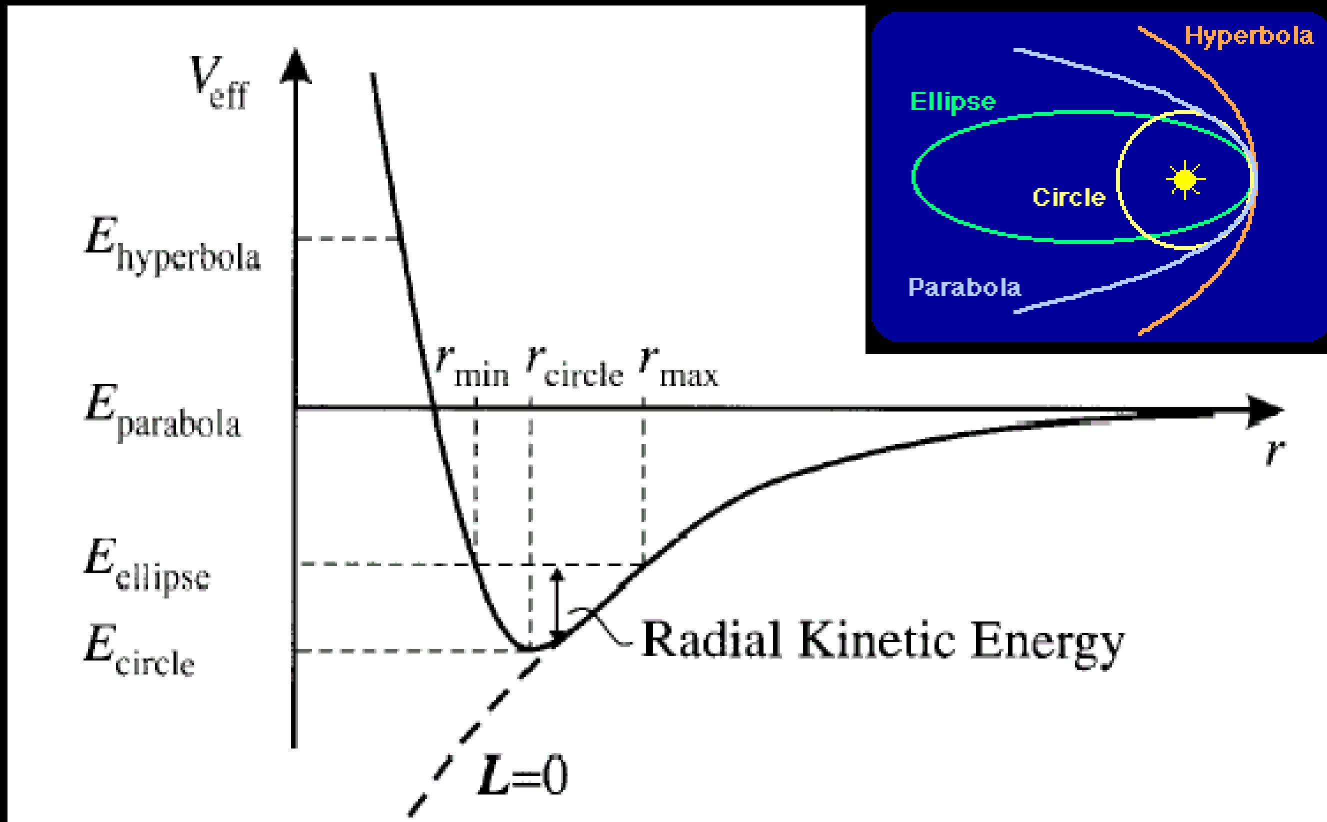
$$t = \frac{2}{3}\left(\sqrt{\frac{r_0^3}{2\mu c^2}} - \sqrt{\frac{r^3}{2\mu c^2}}\right) + \frac{4\mu}{c}\left(\sqrt{\frac{r_0}{2\mu}} - \sqrt{\frac{r}{2\mu}}\right)$$
$$+ \frac{2\mu}{c}\ln\left|\left(\frac{\sqrt{r/(2\mu)}+1}{\sqrt{r/(2\mu)}-1}\right)\left(\frac{\sqrt{r_0/(2\mu)}-1}{\sqrt{r_0/(2\mu)}+1}\right)\right|,$$

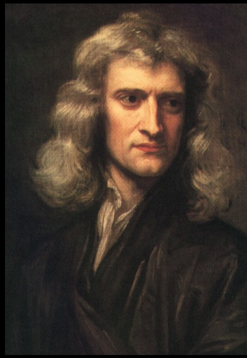
$t = 0$ at $r = r_0$.

$$\tau \to \frac{2}{3}\sqrt{\frac{r_0^3}{2\mu c^2}} \qquad \text{as } r \to 0,$$

$$t \to \infty \qquad \text{as } r \to 2\mu.$$

Potencial efectivo



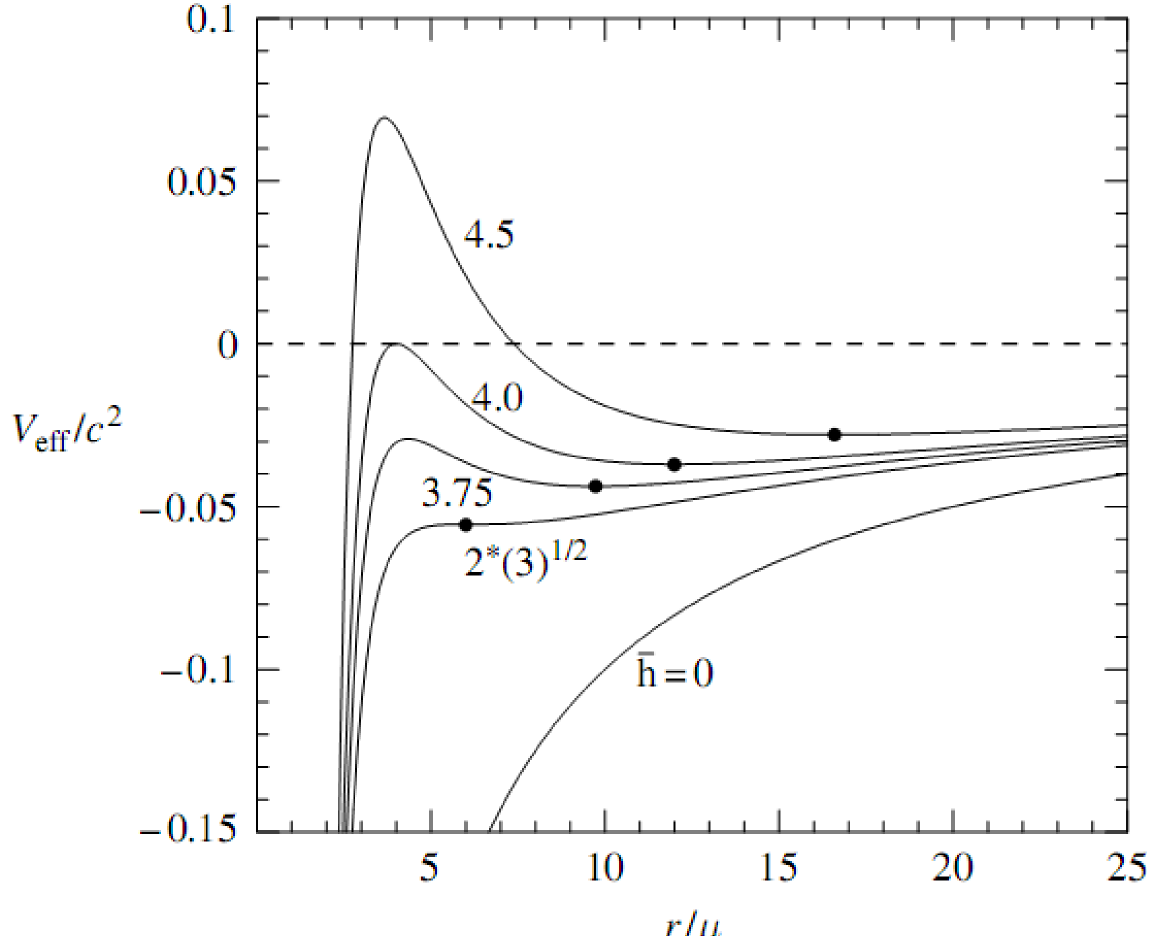


$$\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 = E - V_e(r) \geq 0$$

$$V_e(r) = -\frac{M}{r} + \frac{L^2}{2mr^2}$$

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{\mu c^2}{r} = \frac{c^2}{2}(k^2 - 1),$$

$$V_{\rm eff}(r) = -\frac{\mu c^2}{r} + \frac{h^2}{2r^2} - \frac{\mu h^2}{r^3},$$



$$\bar{h} \equiv h/(c\mu)$$

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Particulas de masa nula

$\theta = \pi/2.$

$$\left(1-\frac{2\mu}{r}\right)\dot{t} = k,$$

$$c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 0,$$

$$r^2\dot{\phi} = h.$$

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0.$$

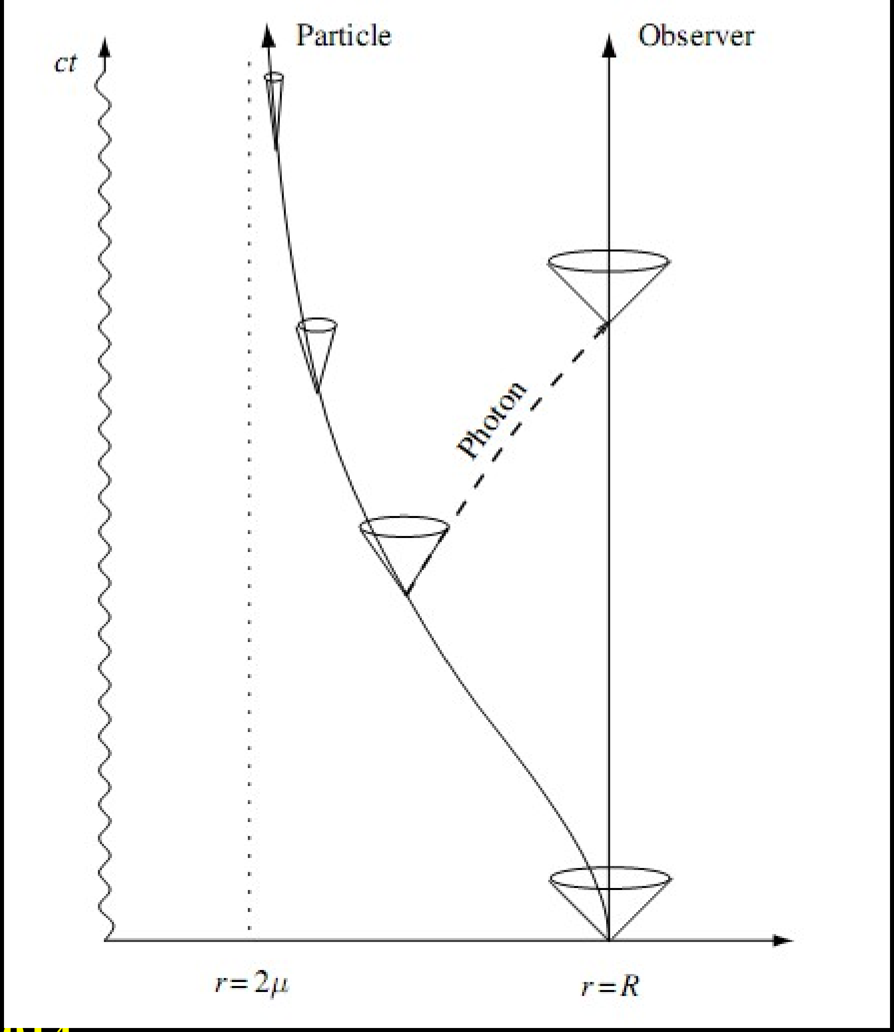
$$\dot{r}^2 + \frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right) = c^2k^2.$$

Movimiento radial → $\dot{\phi} = 0$

$$c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2-\left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2=0,$$

$$\frac{dr}{dt}=\pm c\left(1-\frac{2\mu}{r}\right).$$

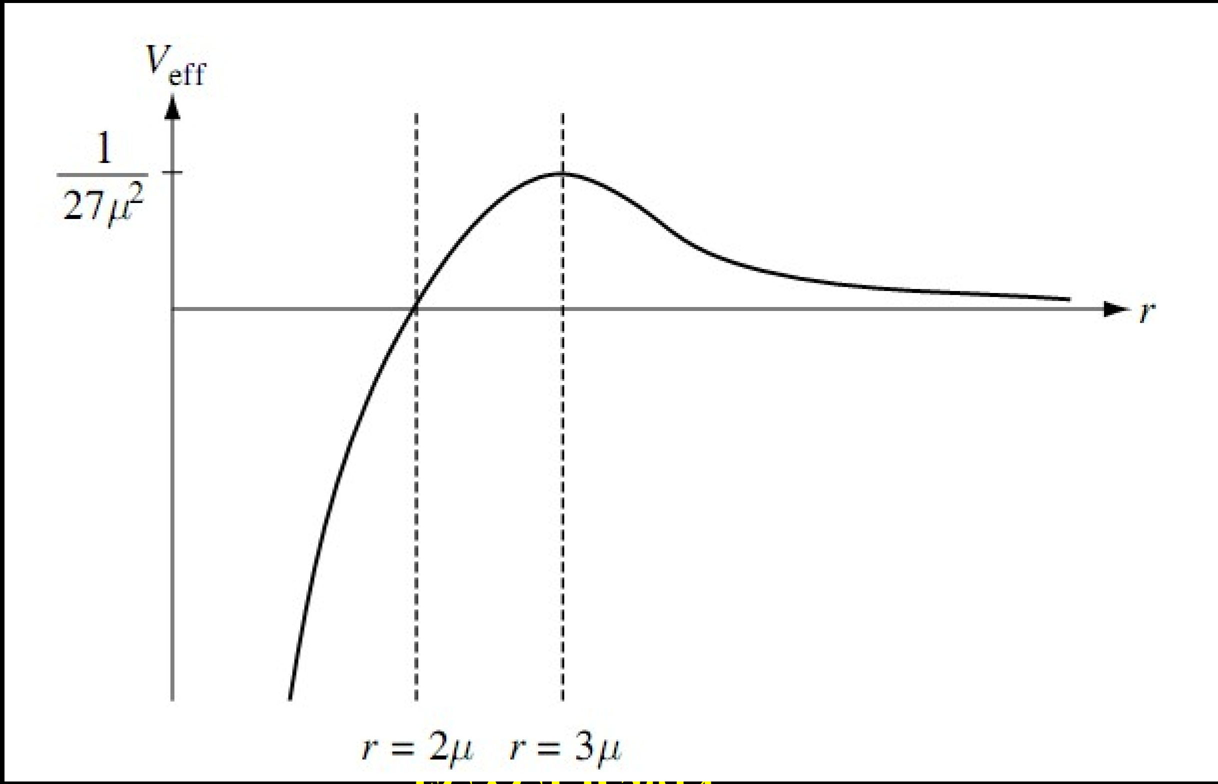
$$ct=-r-2\mu\ln\left|\frac{r}{2\mu}-1\right|+\text{constant}$$



$$\frac{\dot{r}^2}{h^2} + V_{\text{eff}}(r) = \frac{1}{b^2},$$

$$V_{\text{eff}}(r) = \frac{1}{r^2}\left(1 - \frac{2\mu}{r}\right).$$

$$b = h/(ck)$$



Que pasa en *r = 2 M* y en *r = 0* ?

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2.$$

$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{48\mu^2}{r^6},$$

Escalar de Kretschmann

Coordenadas de Eddington-Finkelstein

$$ct = -r - 2\mu \ln \left| \frac{r}{2\mu} - 1 \right| + \text{constant}.$$

Linea de universo de un foton entrante

Nueva coordenada:

$$p = ct + r + 2\mu \ln \left| \frac{r}{2\mu} - 1 \right| ,$$

$$dp = c\, dt + \frac{r}{r - 2\mu}\, dr, \quad ds^2 = \left(1 - \frac{2\mu}{r}\right) dp^2 - 2\, dp\, dr - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Es regular en $0 < r < \infty$

$$ct' \equiv p - r = ct + 2\mu \ln \left| \frac{r}{2\mu} - 1 \right| .$$

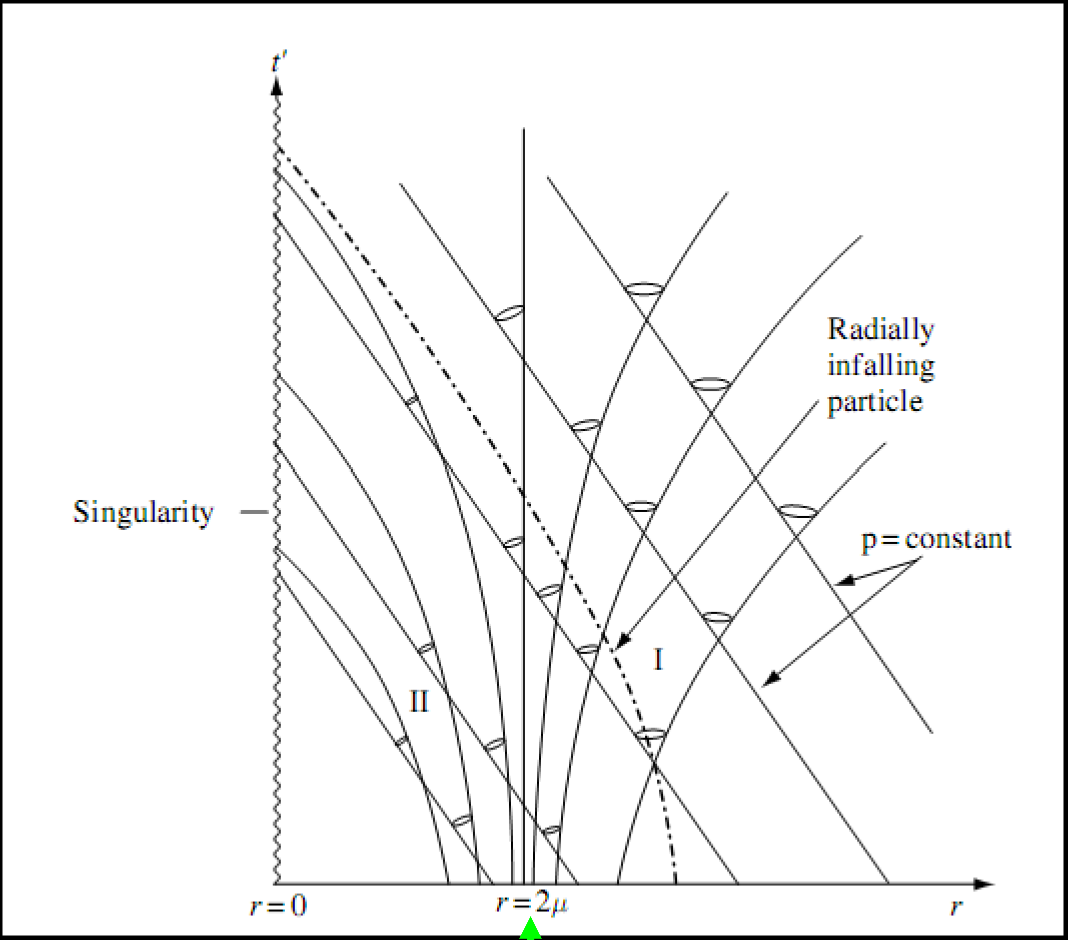
$$ds^2 = c^2 \left(1 - \frac{2\mu}{r}\right) dt'^2 - \frac{4\mu c}{r}\, dt'\, dr - \left(1 + \frac{2\mu}{r}\right) dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Es regular en $0 < r < \infty$

Coordenadas de Eddington-Finkelstein avanzadas

$$ct' = -r + \text{constant}, \qquad \textit{incoming}$$

$$ct' = r + 4\mu \ln \left| \frac{r}{2\mu} - 1 \right| + \text{constant}. \qquad \textit{outgoing}$$



Horizonte

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Como entender um buraco negro de Schwarszchild em termos dos cones de luz?

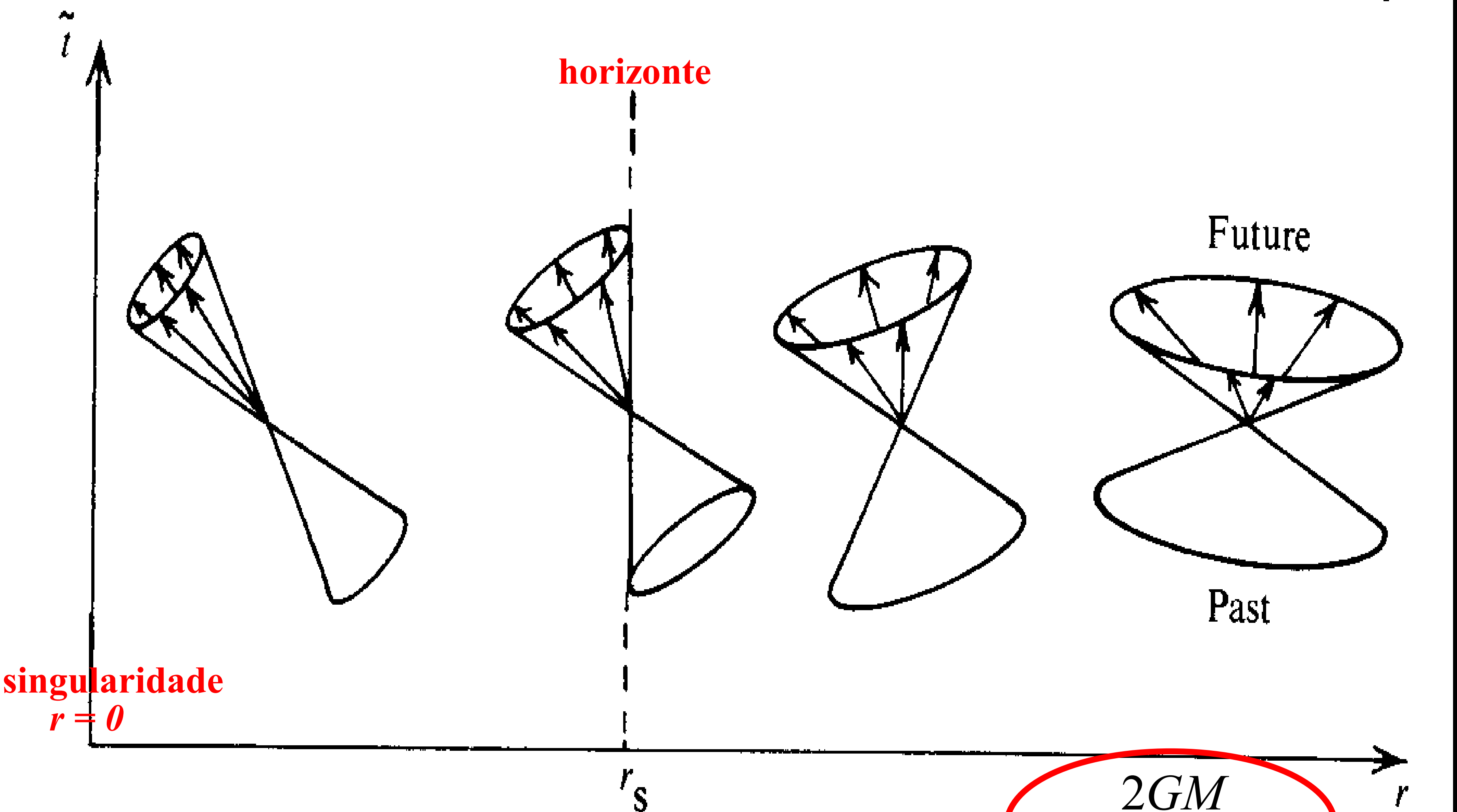
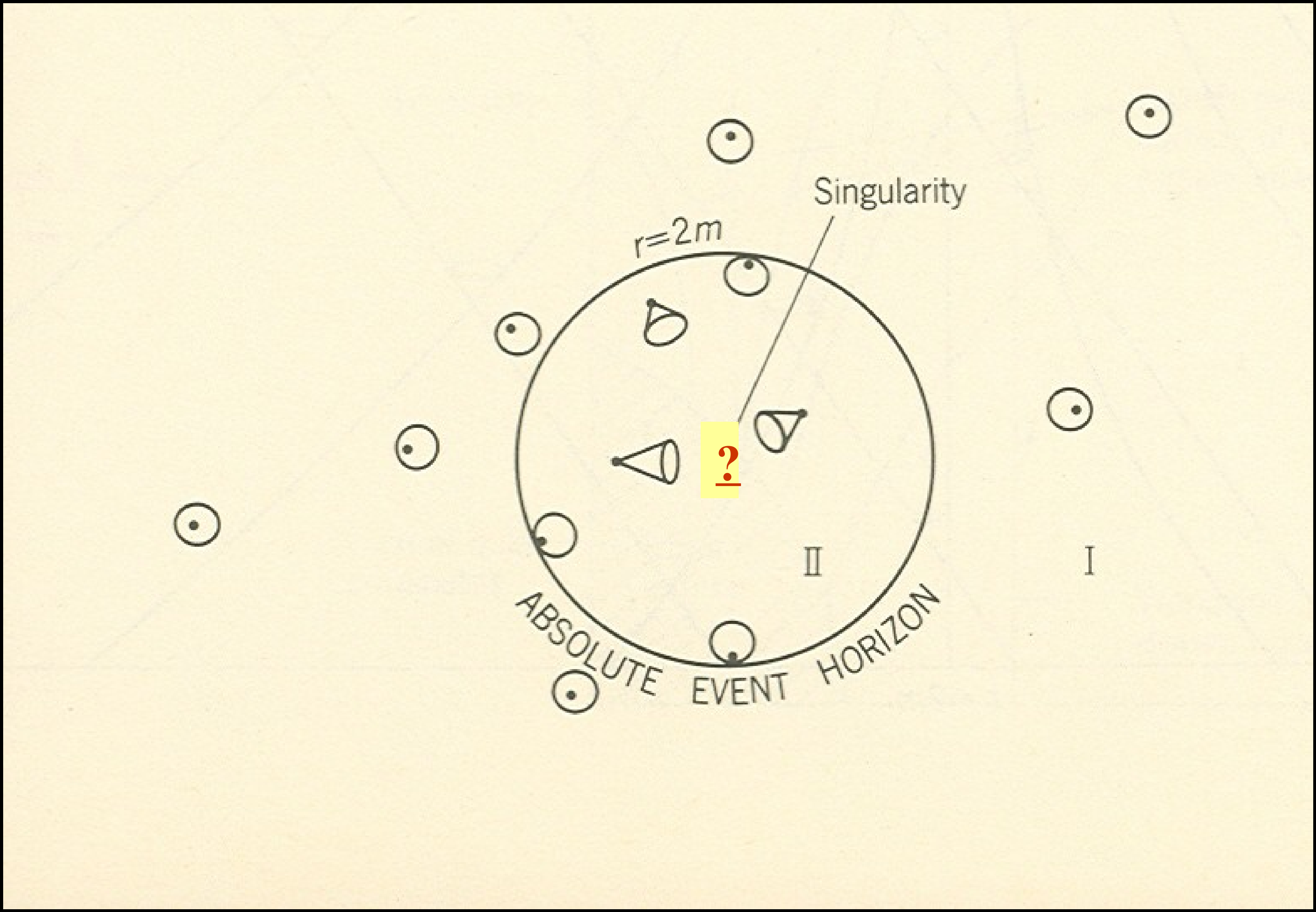


Figure 46. Light cones near a black hole.

$$r_s = \frac{2GM}{c^2}$$



Corte da sol. de Schwarzschild no equador

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$$r = \rho \left(1 + \frac{\mu}{2\rho}\right)^2 ,$$

$$ds^2 = c^2 \left(1 - \frac{\mu}{2\rho}\right)^2 \left(1 + \frac{\mu}{2\rho}\right)^{-2} dt^2 - \left(1 + \frac{\mu}{2\rho}\right)^4 (d\rho^2 + \rho^2\, d\theta^2 + \rho^2 \sin^2\theta\, d\phi^2).$$

Forma isotrópica de la metrica

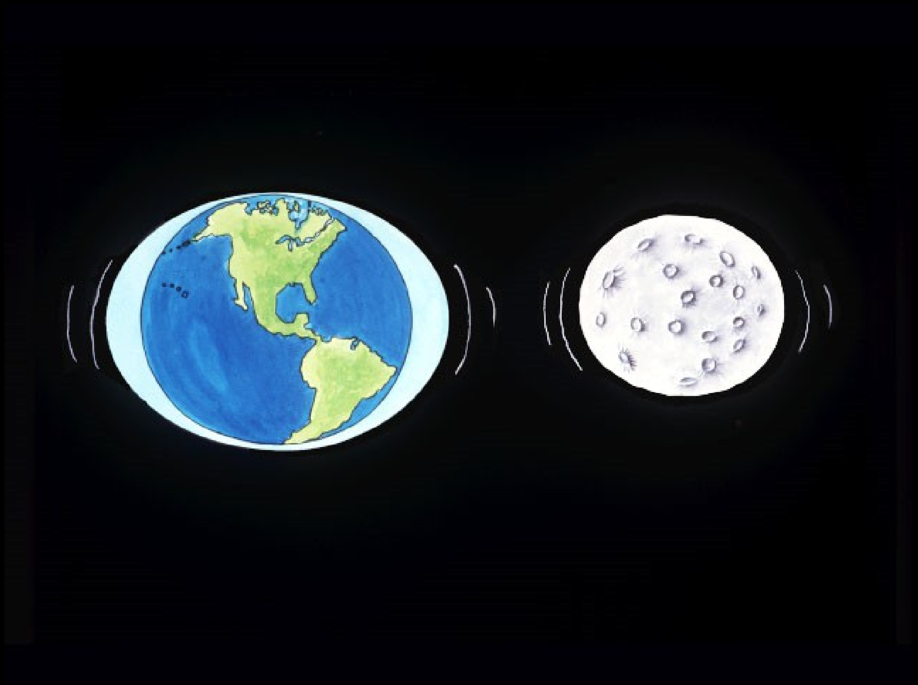
Que pasa em $r = 0$?
Respuesta 1: una singularidad

Punto singular : aquel donde existe alguna cantidad infinita

Ejemplo: campo eléctrico de una carga puntual en $r = 0$

$$\vec{E} = k\frac{q}{r^2}\breve{e}_r$$

Para identificar posibles singularidades podemos utilizar las fuerzas de marea



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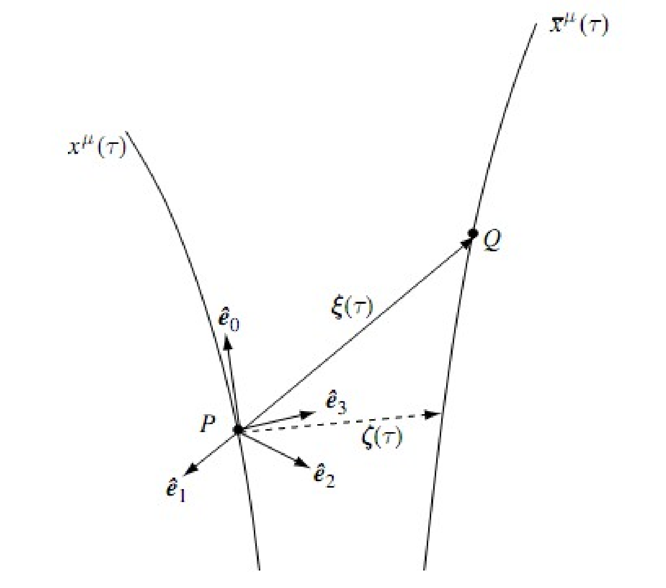
$$(\hat{e}_0)^\mu = \frac{1}{c}u^\mu = \frac{1}{c}\left(1 - \frac{2\mu}{r}\right)^{-1/2}\delta_0^\mu, \qquad (\hat{e}_1)^\mu = \left(1 - \frac{2\mu}{r}\right)^{1/2}\delta_1^\mu,$$

$$(\hat{e}_2)^\mu = \frac{1}{r}\delta_2^\mu, \qquad (\hat{e}_3)^\mu = \frac{1}{r\sin\theta}\delta_3^\mu.$$

$$R^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}\hat{\delta}} \equiv R^\mu{}_{\sigma\nu\rho}(\hat{e}^\alpha)_\mu(\hat{e}_\beta)^\sigma(\hat{e}_\gamma)^\nu(\hat{e}_\delta)^\rho.$$

$$\frac{d^2\xi^{\hat{\alpha}}}{d\tau^2} = c^2 R^{\hat{\alpha}}{}_{\hat{0}\hat{0}\hat{\gamma}}\xi^{\hat{\gamma}},$$

$$\frac{d^2\zeta^{\hat{r}}}{d\tau^2} = +\frac{2\mu c^2}{r^3}\zeta^{\hat{r}}, \qquad \frac{d^2\zeta^{\hat{\theta}}}{d\tau^2} = -\frac{\mu c^2}{r^3}\zeta^{\hat{\theta}}, \qquad \frac{d^2\zeta^{\hat{\phi}}}{d\tau^2} = -\frac{\mu c^2}{r^3}\zeta^{\hat{\phi}}.$$



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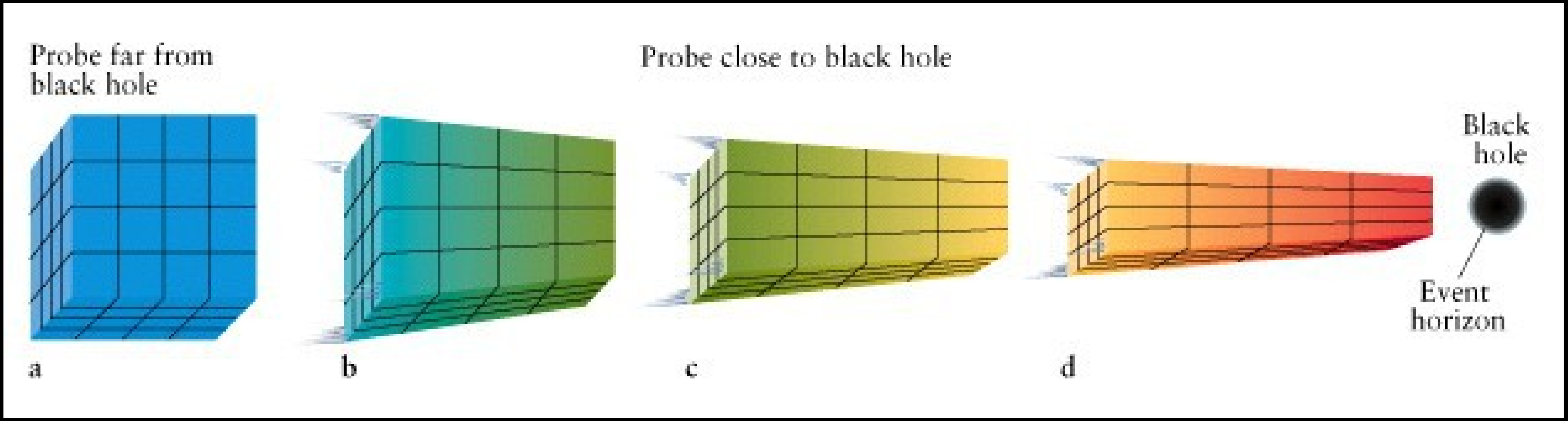

Singularidade = forcas de maré infinitas

(mas note que no horizonte as forças de maré são finitas!)

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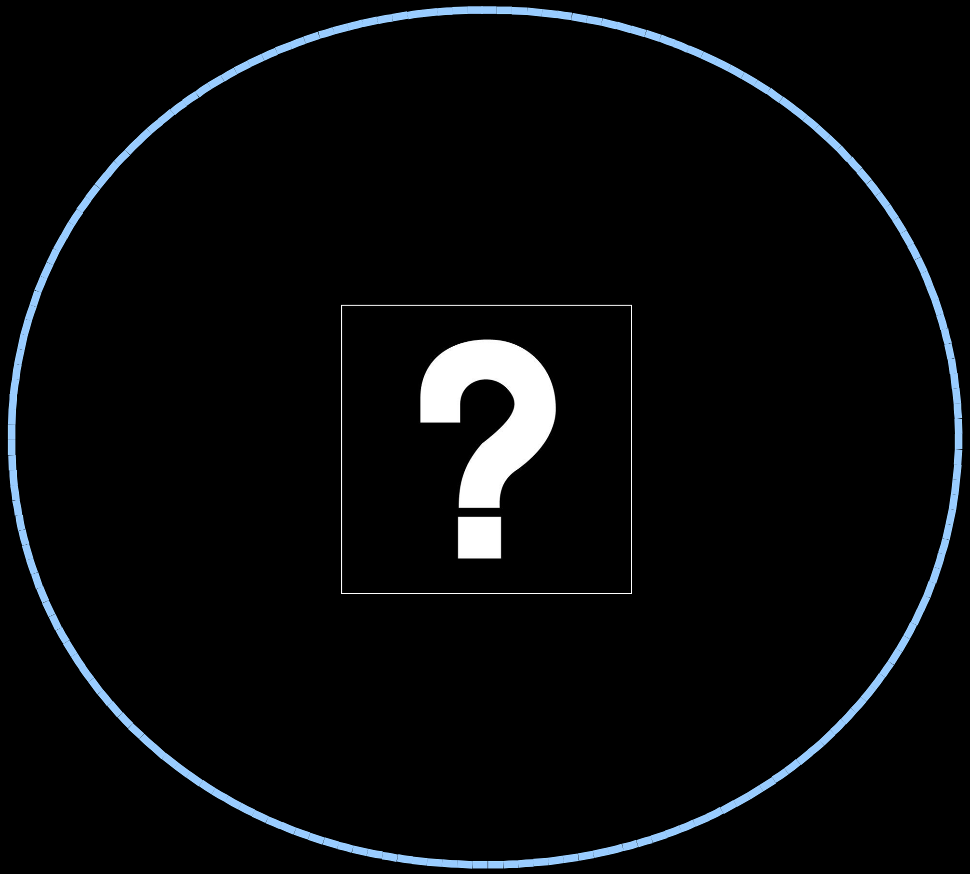
O que há “dentro” do horizonte?
Resposta 2:



Gravitação quântica: não há singularidade?

Brevísima introducción al Maple:

Ver intromaple.mws y orbitasschw.mws

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