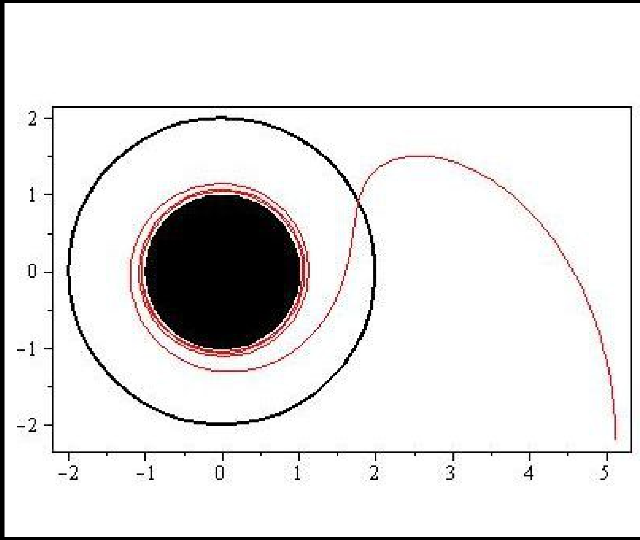

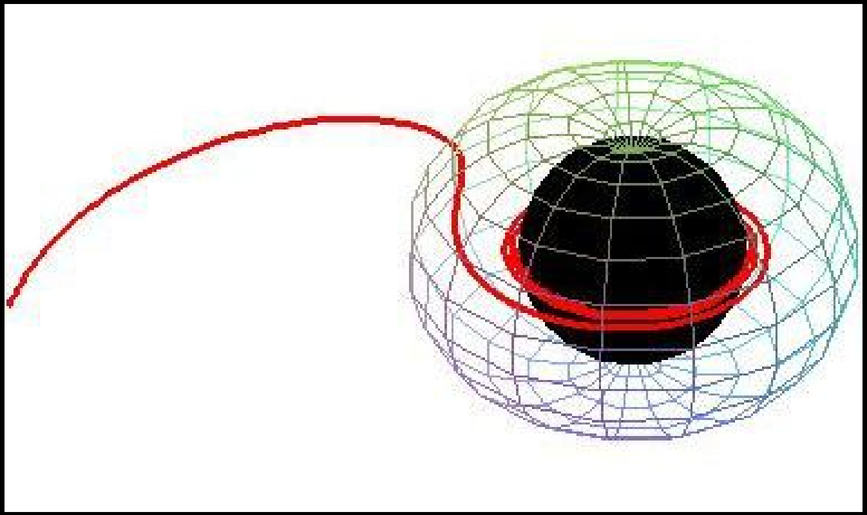

Aplicaciones de GRTensor en Astrofísica y Cosmologia

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FCAGLP 2014

GRTensor

Paquete para el cálculo y manipulación de componentes de tensores y objetos relacionados a estos.

Entre outras cosa, el GRTensor puede

Calcular componentes de tensores (built-in o definidos)

Simplificar expresiones que contengan componentes de tensores

Hacer cambios de coordenadas

Trabajar em el formalismo métrico y en el de tetradas

Determinar "junction conditions" entre dos e-ts

...

Comienzo y sintaxis

```
> readlib ( grii ):
> grtensor():
```

Los tensores se especifican por el nombre y la posición de los índices:

`R(dn,dn)` $(= R_{ab})$, `R(up,dn)` $(= R^{a}{}_{b})$,

`R(bup,bdn)` $(= R^{(a)}{}_{(b)})$,

(componentes em una base)

Especificación de la geometria del espacio-tiempo (en n dim.)

```
makeg()
```

4 posibilidades:

1) Métrica g_{ab}, matriz simétrica $n \times n$

2) Elemento de línea: $ds^2 = ...$

3) Base no holonómica: conjunto de vectores base com producto interno

4) Tetradas nula: formalismo de Newman-Penrose

Bases

$$\{e_{(1)}{}^a = [e_{11}, \ldots, e_{1n}], \quad \ldots \quad e_{(n)}{}^a = [e_{n1}, \ldots, e_{nn}], \}$$

$$\eta^{(a)(b)} = \begin{bmatrix} \eta_{11} & \eta_{12} & \cdots & \eta_{1n} \\ \eta_{12} & \eta_{22} & & \vdots \\ \vdots & & \ddots & \\ \eta_{1n} & \cdots & & \eta_{nn} \end{bmatrix};$$

$$\langle e_{(i)}{}^a, e_{(j)}{}^b \rangle = \eta^{(i)(j)} e_{(i)}{}^a e_{(j)}{}^b = g^{ab}.$$

Cálculos: el comando *grcalc()*

Los comandos que actúan sobre tensores tienen la forma

Nombredelcomando(Nombredeltensor (índices), [opcionales])

Ejemplo: *> grcalc (R (dn, dn), R(up,dn,dn,dn)):*

El Grtensor permite definir nuevos objetos, y tiene una serie de tensores predefinidos:

Ejemplos

`R(dn,dn,dn,dn)`	Riemann tensor, $R^a{}_{bcd} := \frac{\partial \Gamma^a_{bd}}{\partial x^c} - \frac{\partial \Gamma^a_{bc}}{\partial x^d} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc}$
`R(dn,dn)`	Ricci tensor, $R_{ab} := R^c{}_{acb}$
`Ricciscalar`	Ricci scalar, $R := R^a{}_a$
`S(dn,dn)`	trace-free Ricci tensor, $S_{ab} := R_{ab} - \frac{1}{n} R g_{ab}$
`C(dn,dn,dn,dn)`	Weyl tensor, $C_{abcd} := R_{abcd} - \frac{2}{n-2}(g_{a[c}S_{d]b} - g_{b[c}S_{d]a}) - \frac{2}{n(n-1)} R g_{a[c}g_{d]b}$
`Cstar(dn,dn,dn,dn)`	$C^*_{abcd} := \frac{1}{2}\epsilon_{abef}C^{ef}{}_{cd}$
`G(dn,dn)`	Einstein tensor, $G_{ab} := R_{ab} - \frac{1}{2} R g_{ab}$

La secuencia de índices tiene varias posibilidades:

`up, dn`	Índices covariantes/contravariantes
`pup, pdn`	Derivadas parciales
`cup, cdn`	Derivadas covariantes
`bup, bdn`	Índices covar./contrav. De la base
`pbup, pbdn`	Derivadas covariantes em la base
`cbup, cbdn`	

Ejemplos:

`R(dn,dn,pdn)`

`R(up,dn,cdn)`

$$R_{ab,c} := \frac{\partial R_{ab}}{\partial x^c},$$

$$R^a{}_{b;c} := R^a{}_{b,c} - \Gamma^d_{bc} R^a{}_d + \Gamma^a_{bd} R^d{}_c,$$

Operadores

`LieD[v,T(up,dn)]`	Lie derivative,
	$L_v T^a{}_b := v^c \nabla_c T^a{}_b - T^c{}_b \nabla_c v^a + T^a{}_c \nabla_b v^a$
`Box[T(up,dn)]`	d'Alembertian, $\Box T^a{}_b := \nabla^c \nabla_c T^a{}_b$
`Dsq[f(x)]`	$\partial_a f(x) \partial^a f(x)$
`CDsq[f(x)]`	$\nabla_a f(x) \nabla^a f(x)$

`vnorm[v]`	vector norm, $v_a v^a$
`h[v](dn,dn)`	projection tensor, $h_{ab} := g_{ab} - (v_a v^a) v_a v_b$
`acc[v](up)`	acceleration, $a^a := v^a \nabla_a v^b$
`expsc[v]`	expansion scalar, $\Theta := \nabla_a v^a$
`shear[v](dn,dn)`	shear tensor, $\sigma_{ab} := \nabla_{(a} v_{b)} + a_{(a} v_{b)} - \frac{1}{3}\Theta h_{ab}$
`shear[v]`	shear scalar, $\sigma := (\sigma^a{}_b \sigma^b{}_a)^{1/2}$
`vor[v](dn,dn)`	vorticity tensor, $\omega_{ab} := a_{[a} v_{b]} - \nabla_{[a} v_{b]}$
`vor[v](up)`	vorticity vector, $\omega^a := \frac{1}{2}\epsilon^{abcd} v_b \omega_{cd}$
`vor[v]`	vorticity scalar, $\omega := (\omega^a{}_b \omega^b{}_a)^{1/2}$
`RayEqn[v]`	Raychaudhuri's equation, $-R_{ab} v^a v^b = v^a \nabla_a \Theta - \nabla_a a^a + \Theta^2 + (\sigma^2 - \omega^2)$
`E[v](dn,dn)`	electric part of the Weyl tensor, $E_{ab} := C_{acbd} v^c v^d$
`H[v](dn,dn)`	magnetic part of the Weyl tensor, $H_{ab} := C^*_{acbd} v^c v^d$
`KillingTest[v]`	determines if v^a is a Killing vector (see also `?killing`).

Por fin, para calcular las componentes de un tensor

```
> grcalc ( R(up,dn,dn,dn), R(dn,dn), Ricciscalar ):
```

Para mostrar el resultado:

```
> grdisplay ( _ ):
```

```
> grcalcd ( R(up,dn), Ricciscalar ):
```

O *grdisplay* mostra solo los no nulos, respetando las simetrias

```
> grcalc ( R(dn,dn,cdn) ):
```

```
> grcalc ( Box[ R(dn,dn) ] ):
```

Podemos calcular uma componente usando

```
grcalc1 ( R(dn,dn,dn,dn), [r,r,theta,phi] ):
```

Es posible también extraer una componente de un tensor ya calculado:

```
> f(x) := grcomponent ( R(up,dn,dn,dn), [t,r,r,r] ):
```

`grclear ( `*objectSeq*` )`

objectSeq – This argument can be either a sequence of GRTensorII objects or one of the special parameters:

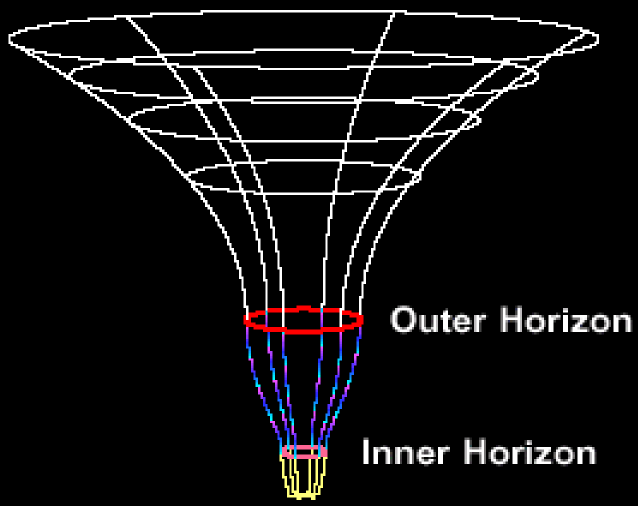
- **results** – Clears all of the GRTensorII objects which have been calculated for the current default spacetime. The default spacetime (ie. metric and/or basis vectors) remains initialized.
- **metric** – Clears all of the objects calculated for the current spacetime, including the metric components `g(dn,dn)`.
- **spacetime** – Clears all of the objects calculated for the current default spacetime as well as the components of the metric and basis vectors, if assigned.

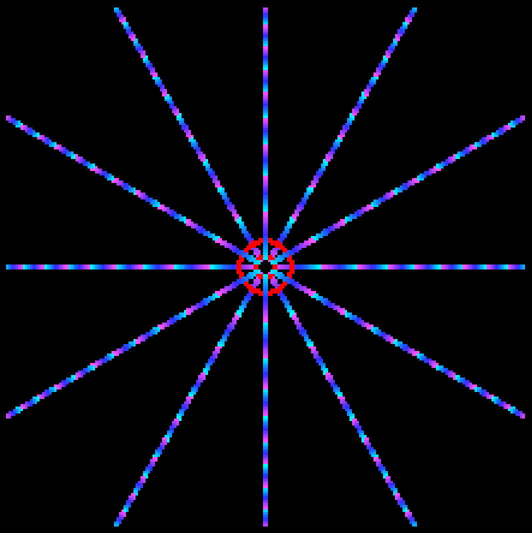
If either of the three special parameters are used as the *objectSeq* argument, then a prompt asks the user to confirm the use of `grclear()`.

Example: `> grclear ( results ):`

ej. 1, 2, 3, 4

Agujero negro de Reissner-Nordstrom *($Q \neq 0$, $L=0$)*





Campo eléctrico

Campo estático com simetria esférica

$$ds^2 = A(r)\,dt^2 - B(r)\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2).$$

$$R_{\mu\nu} = -\kappa T_{\mu\nu}.$$

$$T_{\mu\nu} = -\mu_0^{-1}(F_{\mu\rho}F_\nu{}^\rho - \tfrac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}),$$

$$\nabla_\mu F^{\mu\nu} = 0,$$
$$\nabla_\sigma F_{\mu\nu} + \nabla_\nu F_{\sigma\mu} + \nabla_\mu F_{\nu\sigma} = 0.$$

$$[F_{\mu\nu}] = E(r)\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} F^{\mu\nu}\right) = 0,$$

(válida para cualquier tensor antisimétrico)

$$g = -A(r)B(r)r^4 \sin^2\theta.$$

$$\partial_1 \left(\sqrt{AB}\, r^2 F^{10}\right) = 0.$$

$$F^{10} = g^{1\mu} g^{0\nu} F_{\mu\nu} = g^{11} g^{00} F_{10} = -E/(AB),$$

$$E(r) = \frac{k\sqrt{A(r)B(r)}}{r^2},$$

$$k = Q/(4\pi\epsilon_0 c),$$

$$ds^2 = c^2\left(1 - \frac{2\mu}{r} + \frac{q^2}{r^2}\right)dt^2 - \left(1 - \frac{2\mu}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

$$\mu = GM/c^2$$

$$q^2 = GQ^2/(4\pi\epsilon_0 c^4),$$

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2}.$$

Esta métrica tiene una singularidad en $r = 0$ y dos horizontes dados por

$$r_\pm = \mu \pm (\mu^2 - q^2)^{1/2}.$$

ex. 5

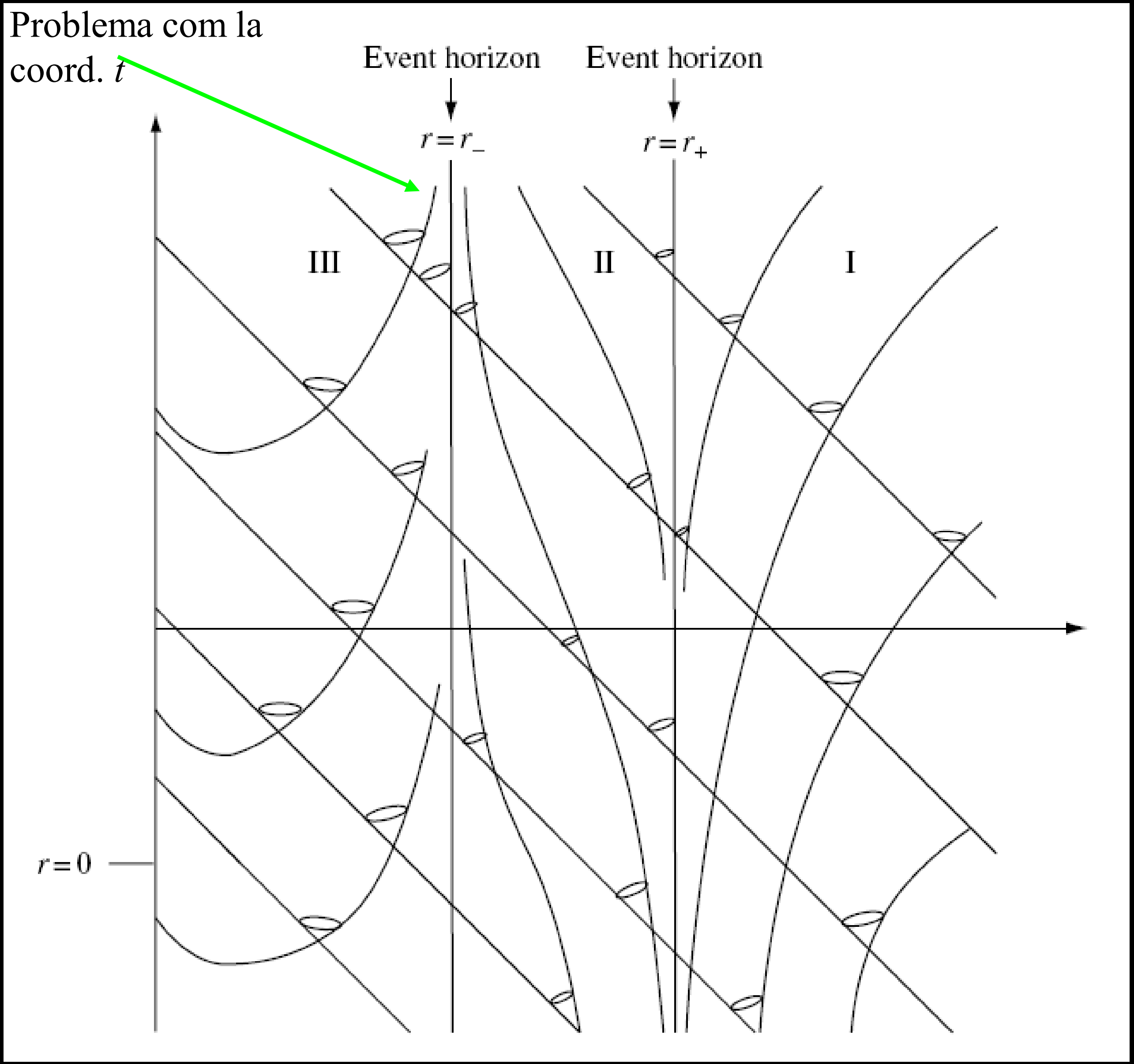
$\mu^2 < q^2$ Singularidad desnuda (no hay horizontes)

$\mu^2 > q^2$ Dos horizontes

$\mu^2 = q^2$ Un horizonte (caso extremo)

Fotones em movimiento radial

$$\frac{dt}{dr} = \pm\frac{1}{c} = \left(1 - \frac{2\mu}{r} + \frac{q^2}{r^2}\right)^{-1} = \pm\frac{1}{c}\frac{r^2}{(r-r_-)(r-r_+)},$$



Problema com la coord. t
Event horizon
Event horizon
r = r_
r = r+
III
II
I
r = 0

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$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{2}{n-2}\left(g_{\alpha[\gamma}R_{\delta]\beta} - g_{\beta[\gamma}R_{\delta]\alpha}\right) + \frac{2}{(n-1)(n-2)}Rg_{\alpha[\gamma}g_{\delta]\beta}.$$

$$g = \varphi^2\left(\left(dx^1\right)^2 + \cdots + \left(dx^n\right)^2\right).$$

(a) Show that the space forms S_k^n are locally conformally flat.

(b) With some help from the literature, show that any 2-dimensional Riemannian manifold is locally conformally flat (isothermal coordinates). In fact, any metric on a closed surface is conformal to a metric of constant curvature. This is called the *uniformization theorem*.

(c) Show that if an Einstein metric is locally conformally flat, then it has constant curvature.

(10) We say that (M, g) admits *orthogonal coordinates* around $p \in M$ if we have coordinates on some neighborhood of p, where

$$|R|^2 = \left|\frac{\text{scal}}{2n\,(n-1)} g \circ g\right|^2 + \left|\left(\text{Ric} - \frac{\text{scal}}{n} \cdot g\right) \circ g\right|^2 + |W|^2 .$$

(b) Show that if we conformally change the metric $\tilde{g} = f \cdot g$, then $\tilde{W} = f \cdot W$.

(c) If (M, g) has constant curvature, then $W = 0$.

(d) If (M, g) is locally conformally equivalent to the Euclidean metric, i.e., locally we can always find coordinates where:

$$g = f \cdot \left(\left(dx^1\right)^2 + \cdots + \left(dx^n\right)^2\right),$$

then $W = 0$. The converse is also true but much harder to prove.

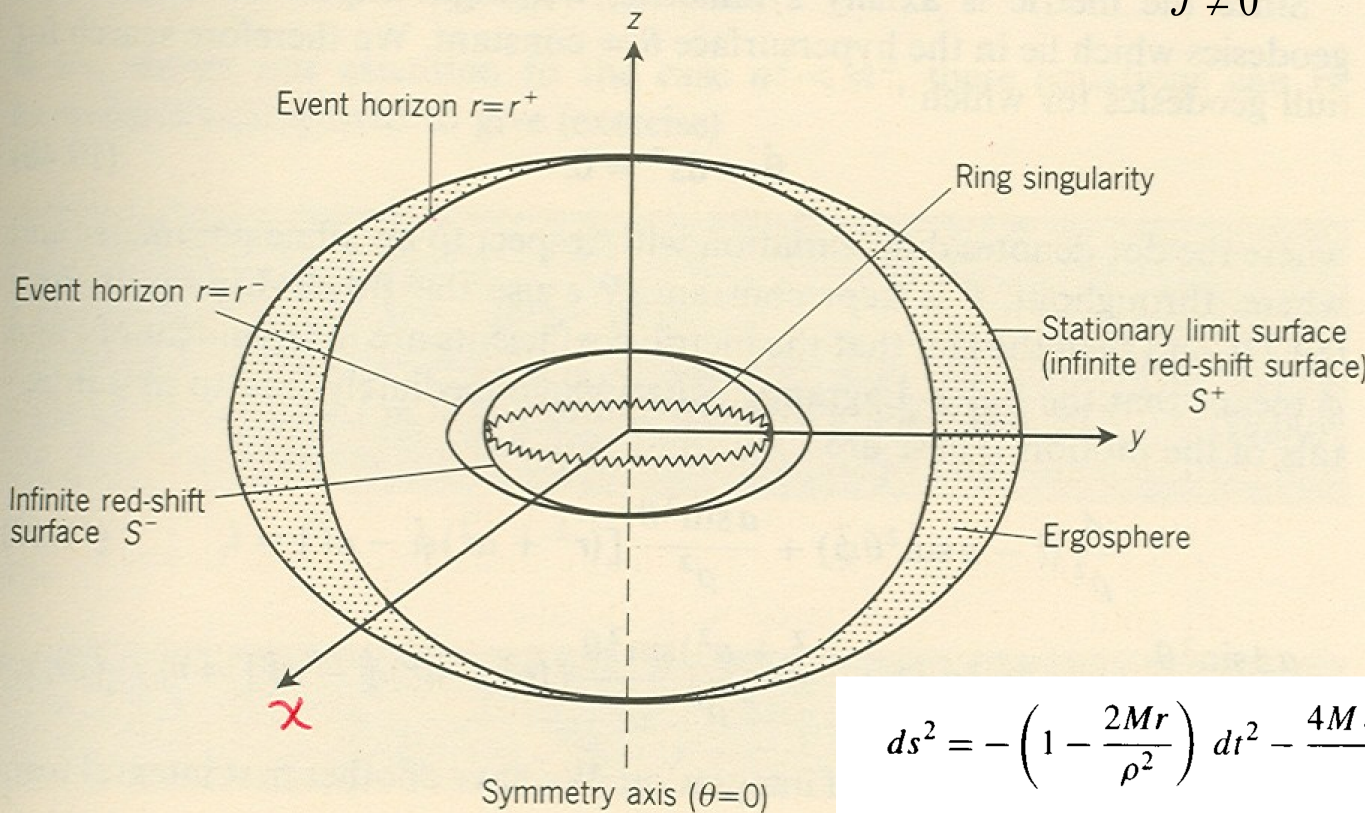
(e) Show that the Weyl tensors for the Schwarzschild metric and the Euguchi-Hanson metrics are not zero.

(f) Show that (M, g) has constant curvature iff $W = 0$ and $\text{Ric} = \frac{\text{scal}}{n}$.

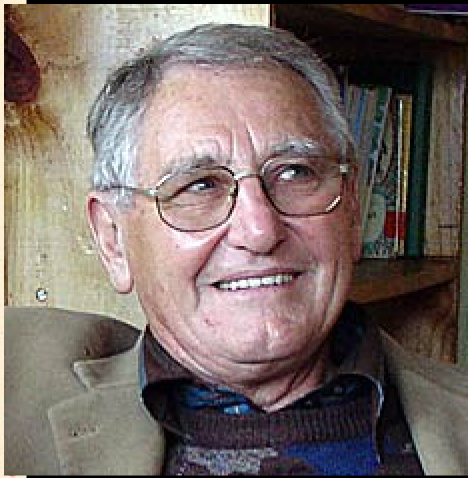
(12) In this problem we shall see that even in dimension 4 the curvature tensor has some very special properties. Throughout we let (M, g) be a 4-dimensional oriented Riemannian manifold. The bi-vectors $\Lambda^2 TM$ come with a natural endomorphism called the Hodge $*$ operator. It is de-

La solución de Kerr

$\vec{J} \neq 0$



Esquema de la sol. de Kerr (1963)



Roy Kerr (1934-)

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4M\,ar\,\sin^2\theta}{\rho^2}\, d\phi dt + \frac{\rho^2}{\Delta}\, dr^2 + \rho^2 d\theta^2$$
$$+ \left(r^2 + a^2 + \frac{2M\,ra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta\, d\phi^2,$$

$$a \equiv J/M, \quad \rho^2 \equiv r^2 + a^2\cos^2\theta, \quad \Delta \equiv r^2 - 2Mr + a^2$$

(simetria axial-limite minkowskiano)

Consideremos la métrica general

$$ds^2 = A\,dt^2 - B(d\phi - \omega\,dt)^2 - C\,dr^2 - D\,d\theta^2,$$

en la que *A, B, C, D*, y ω son función de *r* y θ.

$$g_{tt} = A - B\omega^2, \qquad g_{t\phi} = B\omega, \qquad g_{\phi\phi} = -B, \qquad g_{rr} = -C, \qquad g_{\theta\theta} = -D,$$

$$g^{rr} = -1/C, \qquad g^{\theta\theta} = -1/D.$$

$$g^{tt} = \frac{g_{\phi\phi}}{|\mathsf{G}|} = \frac{1}{A}, \qquad g^{t\phi} = -\frac{g_{t\phi}}{|\mathsf{G}|} = \frac{\omega}{A}, \qquad g^{\phi\phi} = \frac{g_{tt}}{|\mathsf{G}|} = \frac{B\omega^2 - A}{AB}.$$

$$|\mathsf{G}| = g_{tt}g_{\phi\phi} - (g_{t\phi})^2 = -AB.$$

1) Arrastre de los sistemas inerciales

Los e-t descriptos por esta métrica tienen dos características particulares:

1) La métrica no depende de $\phi \rightarrow p_\phi$ se conserva.

$$p^\phi = g^{\phi\alpha} p_\alpha = g^{\phi\phi} p_\phi + g^{\phi t} p_t,$$

$$p^t = g^{t\alpha} p_\alpha = g^{tt} p_t + g^{t\phi} p_\phi.$$

Consideremos una partícula con $p_\phi = 0$. Recordando que

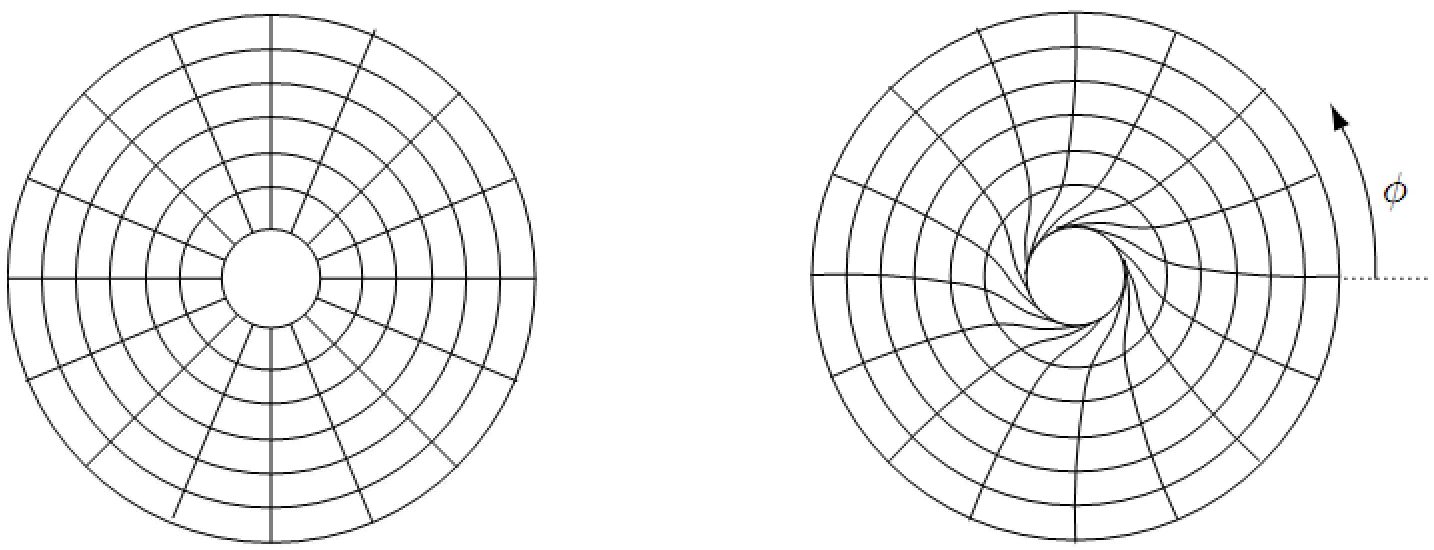
$$p^t = m\, \mathrm{d}t/\mathrm{d}\tau, \qquad p^\phi = m\, \mathrm{d}\phi/\mathrm{d}\tau,$$

tenemos que sobre la trayectoria de la particula

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{p^\phi}{p^t} = \frac{g^{\phi t}}{g^{tt}} \equiv \omega(r, \theta).$$

Una partícula con $p_\Phi = 0$ es "arrastrada " por el campo gravitacional y adquiere una velocidad angular en el caso de un e-t descrito por la métrica

$$ds^2 = A\,dt^2 - B(d\phi - \omega\,dt)^2 - C\,dr^2 - D\,d\theta^2,$$



$t = 0$ $\quad$ $t > 0$

2) Superficies limite estacionarias

Consideremos dos fotones emitidos desde un punto dado em las direcciones $\pm\phi$

$$g_{tt}\, dt^2 + 2g_{t\phi}\, dt\, d\phi + g_{\phi\phi}\, d\phi^2 = 0,$$

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \left[\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}\right]^{1/2}.$$

ω

Suponiendo que g_{tt} es positivo en el punto de emisión, $d\phi/dt$ es positivo (negativo) para un foton emitido em la dirección $+\phi$ $(-\phi)$.

Caso $g_{tt}(r,\theta) = 0,$

$$\frac{d\phi}{dt} = -\frac{2g_{t\phi}}{g_{\phi\phi}} = 2\omega$$

(en la dirección de la rotacion)

$$\frac{d\phi}{dt} = 0.$$

(en la dirección contraria a la rotación)

Inicialmente el foton emitido en la dirección contraria a la rotación no se mueve.

Cualquier particula de masa no nula girará com la fuente, aunque tenga momento angular grande.

Dentro de la superficie definida por $g_{tt} = 0$ ninguna particula puede permanecer en reposo:

$$[u^\mu] = (u^t, 0, 0, 0).$$

cuadrivelocidad de uma particula em reposo en (r, θ, Φ)

La cuadrivelocidad tiene que satisfacer

$$\boldsymbol{u} \cdot \boldsymbol{u} = g_{tt}(u^t)^2 = c^2,$$

pero esto no es posible cuando $g_{tt} < 0$

Otra propiedad de la superficie $g_{tt} = 0$:

A partir de

$$\frac{\nu_R}{\nu_E} = \left[\frac{g_{tt}(A)}{g_{tt}(B)}\right]^{1/2}$$

que dá el redshift de un foton emitido en A y recibido em B en un e-t estacionario, vemos que $\nu_R \rightarrow 0$ cuando $g_{tt}(A) \rightarrow 0$

La superficie $g_{tt}(r, \theta) = 0$ es uma superficie de redshift infinito.

La métrica de Kerr

$$ds^2 = A\,dt^2 - B(d\phi - \omega\,dt)^2 - C\,dr^2 - D\,d\theta^2,$$

$$R_{\mu\nu} = 0.$$

$$ds^2 = c^2\left(1 - \frac{2\mu r}{\rho^2}\right)dt^2 + \frac{4\mu acr\sin^2\theta}{\rho^2}\,dt\,d\phi - \frac{\rho^2}{\Delta}\,dr^2 - \rho^2\,d\theta^2 - \left(r^2 + a^2 + \frac{2\mu ra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta\,d\phi^2,$$

(t, r, θ, ϕ) Coordenadas de Boyer-Lindqist

$$ds^2 = c^2\left(1 - \frac{2\mu r}{\rho^2}\right) dt^2 + \frac{4\mu a c r \sin^2\theta}{\rho^2}\, dt\, d\phi - \frac{\rho^2}{\Delta}\, dr^2 - \rho^2\, d\theta^2$$
$$- \left(r^2 + a^2 + \frac{2\mu r a^2 \sin^2\theta}{\rho^2}\right) \sin^2\theta\, d\phi^2,$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2\mu r + a^2.$$

$$ds^2 = \frac{\Delta - a^2\sin^2\theta}{\rho^2}\, c^2\, dt^2 + \frac{4\mu a r \sin^2\theta}{\rho^2}\, c\, dt\, d\phi$$
$$- \frac{\rho^2}{\Delta}\, dr^2 - \rho^2\, d\theta^2 - \frac{\Sigma^2 \sin^2\theta}{\rho^2}\, d\phi^2.$$

$$\Sigma^2 = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$$

$$ds^2 = \frac{\rho^2\Delta}{\Sigma^2}c^2\,dt^2 - \frac{\Sigma^2\sin^2\theta}{\rho^2}(d\phi - w\,dt)^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2\,d\theta^2,$$

$$\omega = 2\mu c r a/\Sigma^2.$$

Límites:

$a \to 0,$

$$ds^2 \to c^2\left(1 - \frac{2\mu}{r}\right)dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2.$$

$$\mu = GM/c^2,$$

Límite ω << 1:

$$J = Mac.$$

Note que las coordenadas de Boyer-Lindquist no son iguales a las de Schwarszchild. Em particular, las superficies t = cte., r = cte. no tienen la métrica de uma 2-esfera.

$$ds^2 = c^2\left(1 - \frac{2\mu r}{\rho^2}\right) dt^2 + \frac{4\mu a c r \sin^2\theta}{\rho^2}\, dt\, d\phi - \frac{\rho^2}{\Delta}\, dr^2 - \rho^2\, d\theta^2 - \left(r^2 + a^2 + \frac{2\mu r a^2 \sin^2\theta}{\rho^2}\right) \sin^2\theta\, d\phi^2,$$

$\mu \to 0,$ deberia ser Minkowski:

$$ds^2 = c^2\, dt^2 - \frac{\rho^2}{r^2 + a^2}\, dr^2 - \rho^2\, d\theta^2 - (r^2 + a^2)\sin^2\theta\, d\phi^2.$$

que es la métrica de Minkowski en coordenadas *(r , θ, Φ)* relacionadas con las cartesianas por

$$x = \sqrt{r^2 + a^2}\sin\theta\cos\phi,$$
$$y = \sqrt{r^2 + a^2}\sin\theta\sin\phi,$$
$$z = r\cos\theta,$$

En coordenadas de Eddington-Finkelstein avanzadas

$$\mathrm{d}s^2 = \left(1 - \frac{2mr}{\rho^2}\right)\mathrm{d}v^2 - 2\,\mathrm{d}v\,\mathrm{d}r + \frac{2mr}{\rho^2}(2a\sin^2\theta)\,\mathrm{d}v\,\mathrm{d}\bar{\phi} + 2a\sin^2\theta\,\mathrm{d}r\,\mathrm{d}\bar{\phi}$$
$$- \rho^2\,\mathrm{d}\theta^2 - \left((r^2 + a^2)\sin^2\theta + \frac{2mr}{\rho^2}(a^2\sin^4\theta)\right)\mathrm{d}\bar{\phi}^2, \qquad (19.22)$$

$$\mathrm{d}v = d\bar{t} + \mathrm{d}r = \mathrm{d}t + \frac{2mr + \Delta}{\Delta}\mathrm{d}r$$

$$\mathrm{d}\bar{\phi} = \mathrm{d}\phi + \frac{a}{\Delta}\mathrm{d}r$$

$$ds^2 = d\bar{t}^2 - dx^2 - dy^2 - dz^2 - \frac{2mr^3}{r^4 + a^2 z^2}\left(d\bar{t} + \frac{r}{a^2 + r^2}(x\,dx + y\,dy) + \frac{a}{a^2 + r^2}(y\,dx - x\,dy) + \frac{z}{r}\,dz\right)^2,$$

$$r^4 - r^2(x^2 + y^2 + z^2 - a^2) - a^2 z^2 = 0.$$

Forma de Kerr-Schild

$$\begin{aligned}
\bar{t} &= v - r, \\
x &= r\sin\theta\cos\phi + a\sin\theta\sin\phi, \\
y &= r\sin\theta\sin\phi - a\sin\theta\cos\phi, \\
z &= r\cos\theta.
\end{aligned}$$

KS $\rightarrow$ EFA

Estructura de la solución de Kerr

Singularidades y horizontes

$$ds^2 = c^2 \left(1 - \frac{2\mu r}{\rho^2}\right) dt^2 + \frac{4\mu acr \sin^2\theta}{\rho^2} dt\, d\phi - \frac{\rho^2}{\Delta} dr^2 - \rho^2\, d\theta^2 - \left(r^2 + a^2 + \frac{2\mu r a^2 \sin^2\theta}{\rho^2}\right) \sin^2\theta\, d\phi^2,$$

$$\rho = 0 \qquad \Delta = 0.$$

$$\rho^2 = r^2 + a^2 \cos^2\theta = 0, \qquad r = 0, \qquad \theta = \pi/2.$$

Singularidad em forma de anillo

$\Delta = 0.$ $\longrightarrow$ $r_{\pm} = \mu \pm (\mu^2 - a^2)^{1/2}$

Veremos despues que estas superficies son horizontes, con

$$a^2 < \mu^2.$$

Tales superficies son axialmente simétricas con geometria dada por

$$d\sigma^2 = \rho_{\pm}^2\, d\theta^2 + \left(\frac{2\mu r_{\pm}}{\rho_{\pm}}\right)^2 \sin^2\theta\, d\phi^2,$$

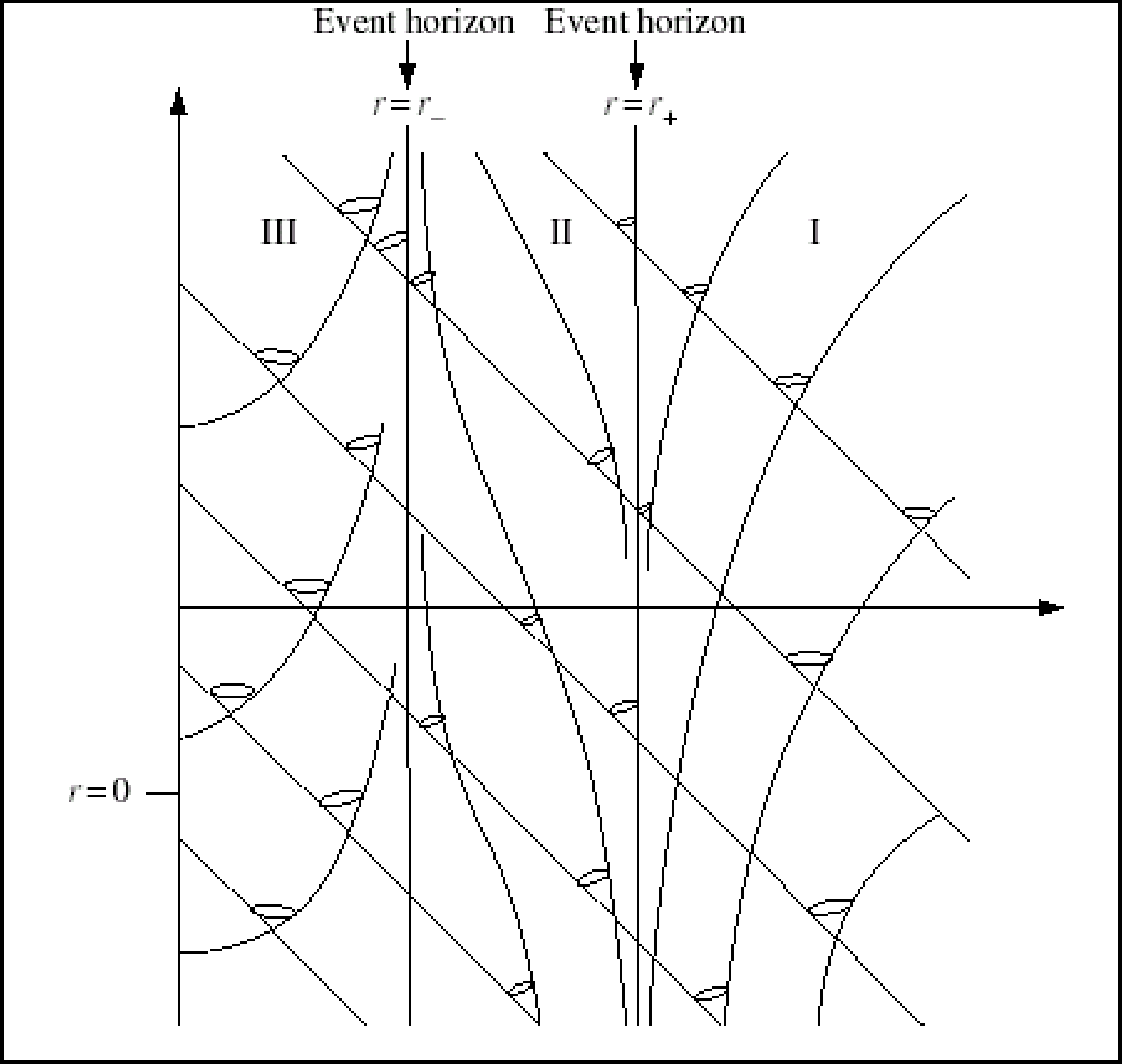
$a^2 > \mu^2$ $\rightarrow$ "singularidad desnuda"

Las superficies límite estacionarias y de redshif infinito (S^+ y S^-) estan dadas por

$g_{tt}(r, \theta) = 0 \rightarrow$

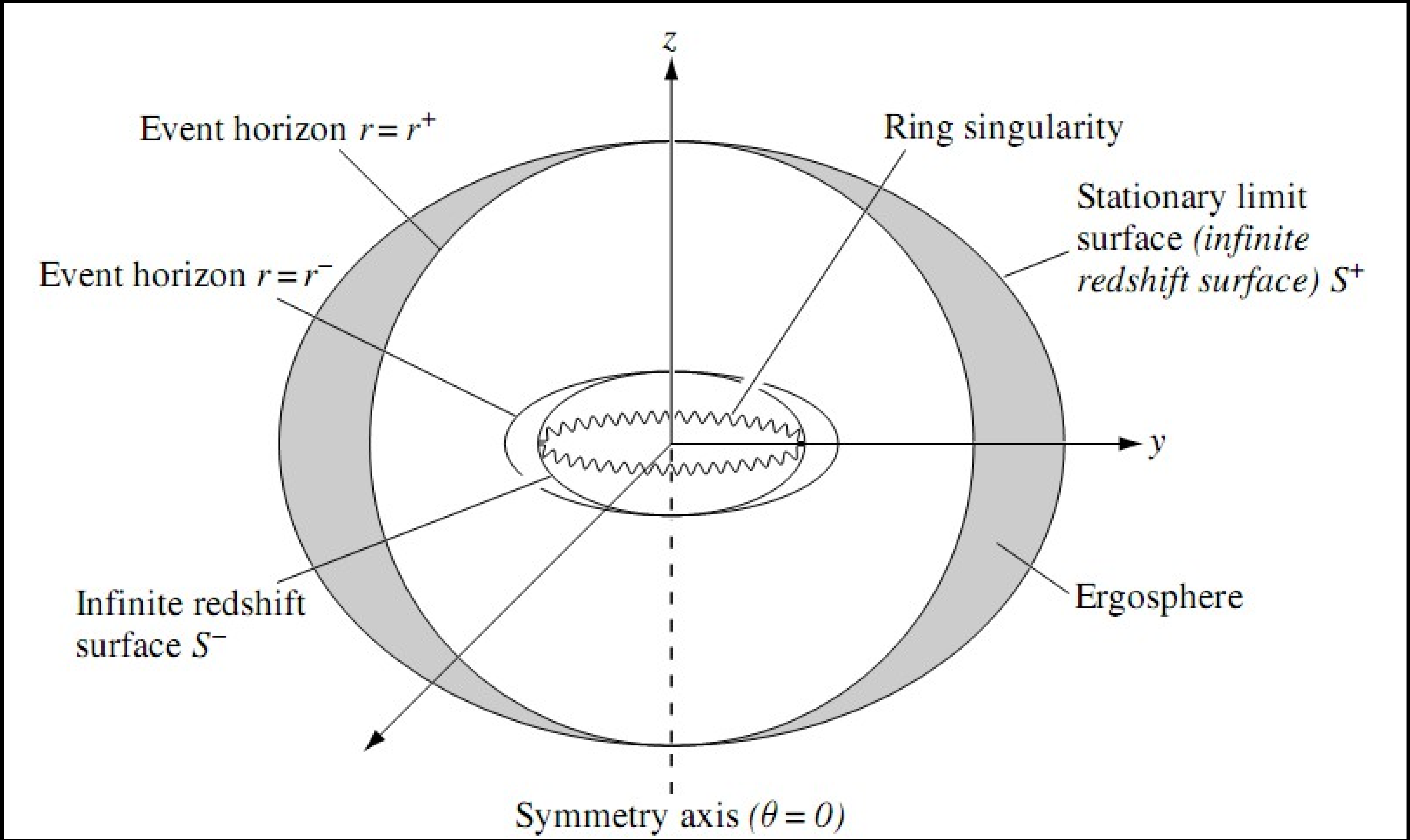
$$r_{S\pm} = \mu \pm \left(\mu^2 - a^2 \cos^2\theta\right)^{1/2}.$$

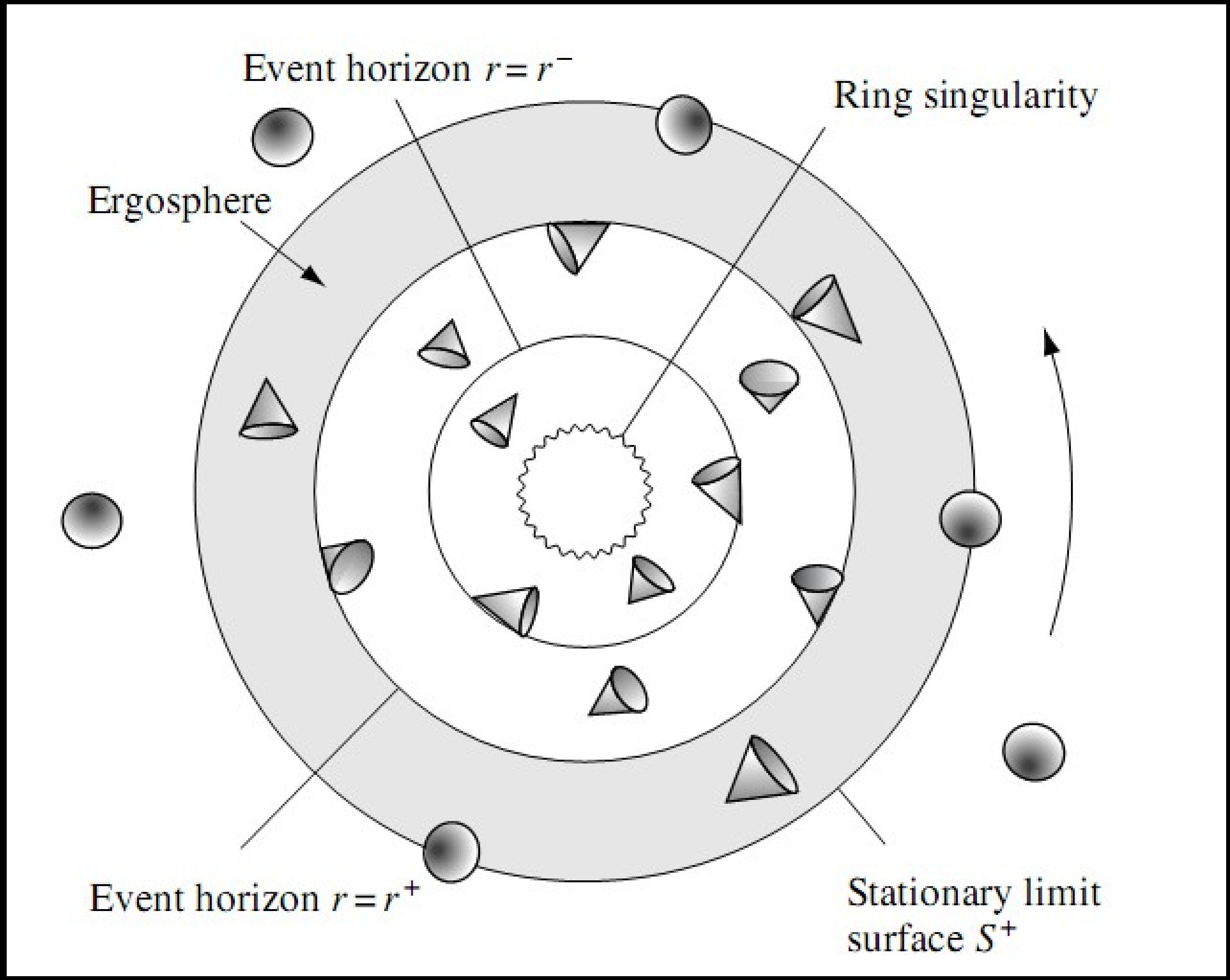
$$d\sigma^2 = \rho^2_{S\pm}\, d\theta^2 + \left[\frac{2\mu r_{S\pm}(2\mu r_{S\pm} + 2a^2 \sin^2\theta)}{\rho^2_{S\pm}}\right] \sin^2\theta\, d\phi^2,$$



Conos de luz em la geometria de Kerr (coords. de EF avanzadas)

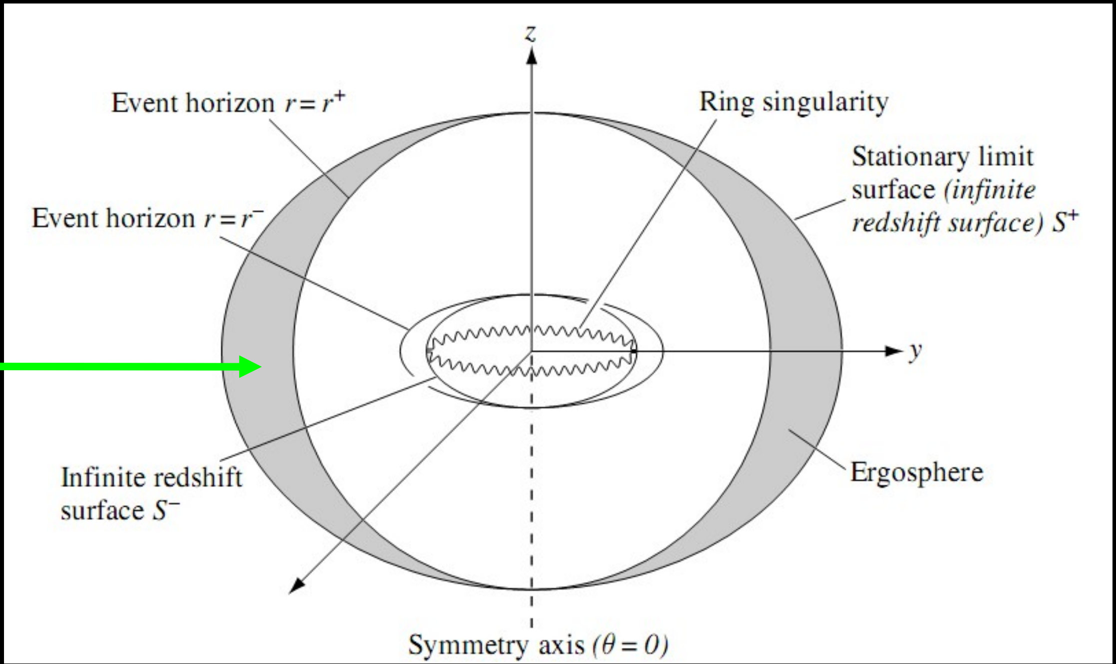
FCAGLP 2014


z
Event horizon $r = r^+$
Ring singularity
Stationary limit surface (infinite redshift surface) S^+
Event horizon $r = r^-$
y
Infinite redshift surface S^-
Ergosphere
Symmetry axis ($\theta = 0$)


Event horizon $r = r^-$
Ring singularity
Ergosphere
Event horizon $r = r^+$
Stationary limit
surface S^+

La ergosfera (*ergo* = trabajo)

$g_{tt} < 0$



afuera del horizonte externo

$g_{tt} < 0$ → no es posible permanecer em un punto fijo, pero sí es posible orbitar con r y θ constantes, em el mismo sentido del agujero negro:

$$[u^\mu] = u^t(1, 0, 0, \Omega),$$

$$\Omega = d\phi/dt$$

$$\boldsymbol{u}\cdot\boldsymbol{u} = g_{\mu\nu}u^{\mu}u^{\nu} = c^2$$

$$[u^{\mu}] = u^t(1, 0, 0, \Omega),$$

$$g_{tt}(u^t)^2 + 2g_{t\phi}u^t u^{\phi} + g_{\phi\phi}(u^{\phi})^2 = (u^t)^2(g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2) = c^2.$$

$$g_{\phi\phi}\Omega^2 + 2g_{t\phi}\Omega + g_{tt} > 0.$$

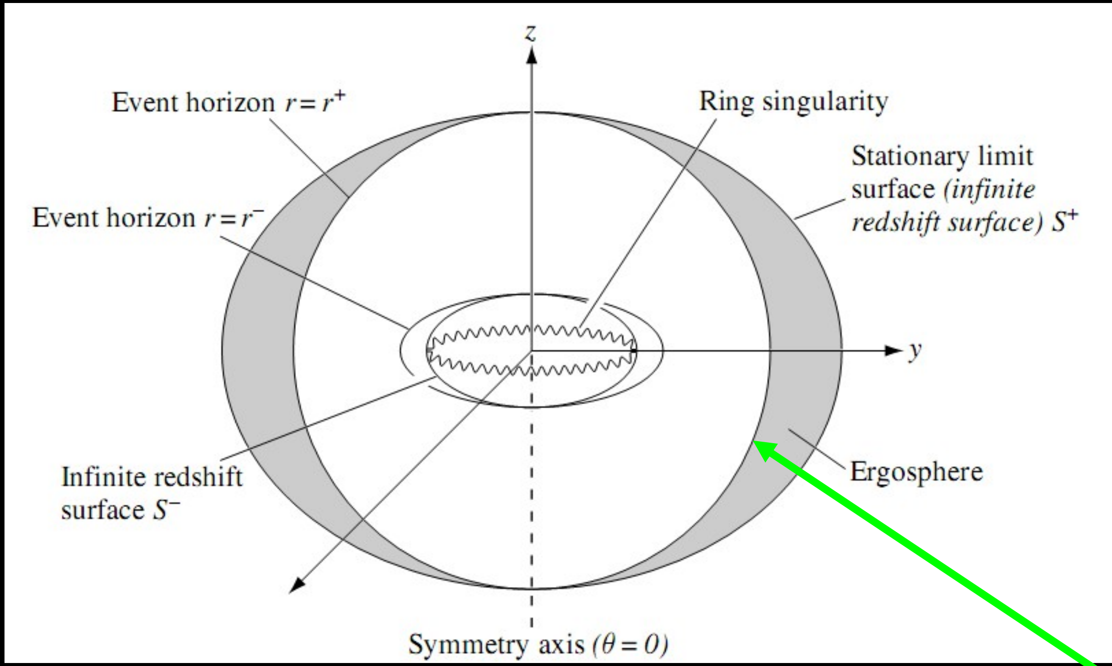
$$\Omega_- < \Omega < \Omega_+$$

$$\Omega_{\pm} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \left[\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}\right]^{1/2} = \omega \pm \left(\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}\right)^{1/2}$$

Dos casos especiales: (a) $g_{tt} = 0$, y (b)

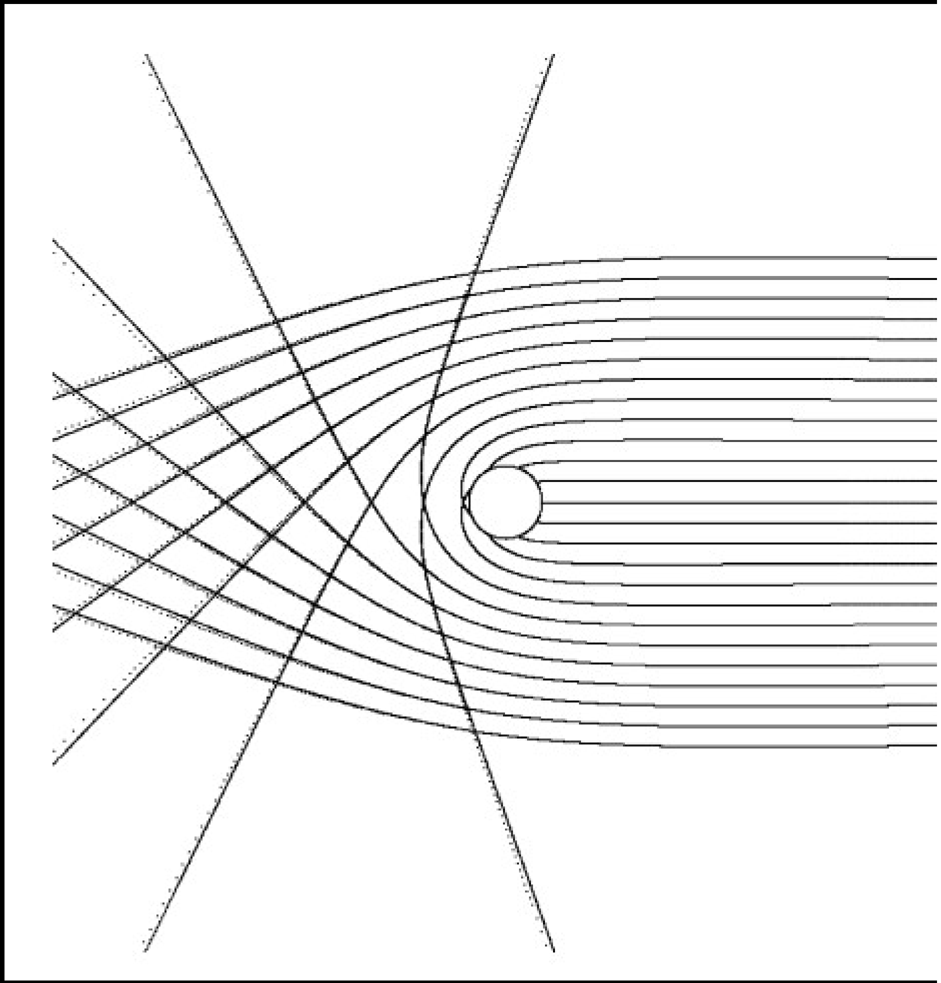
$$\omega^2 = g_{tt}/g_{\phi\phi}, \longrightarrow \Omega_\pm = \omega.$$

$$\Delta = 0, \longrightarrow r = r_+$$

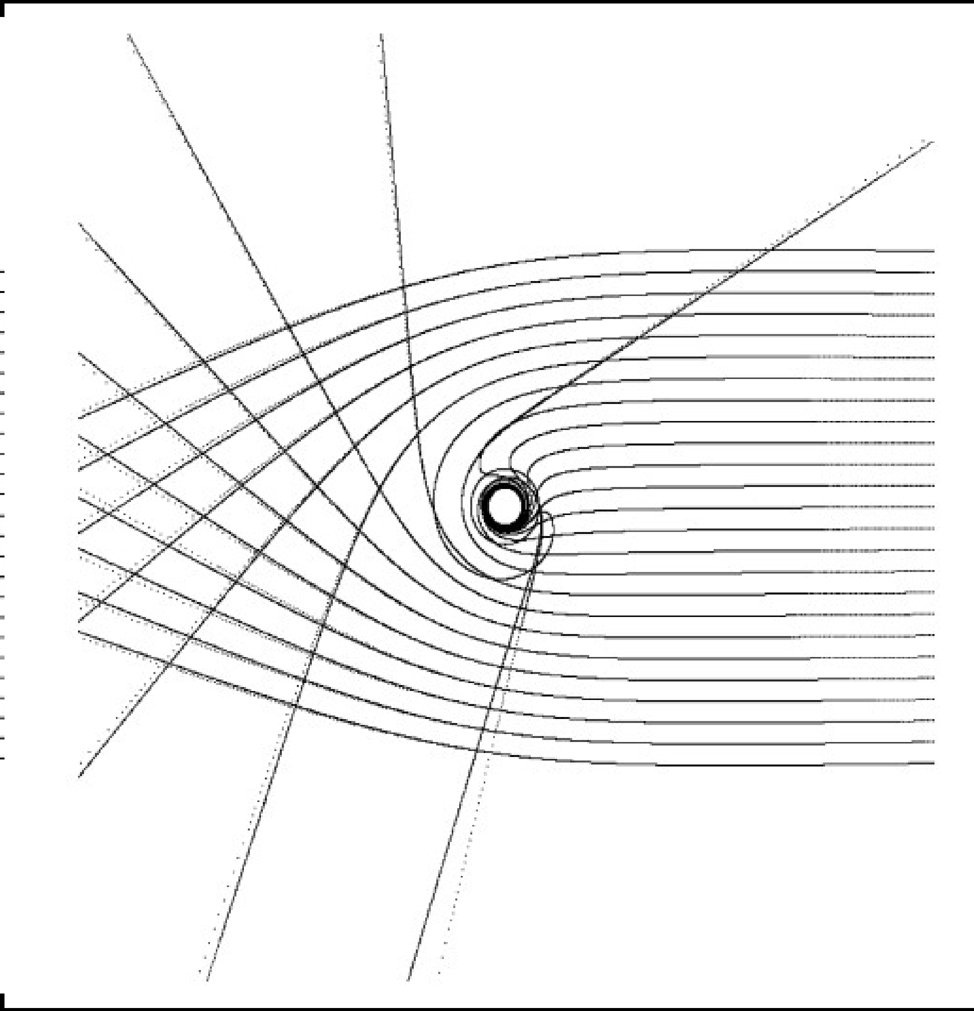


$$\Omega_H \equiv \omega(r_+, \theta) = \frac{ac}{2\mu r_+},$$

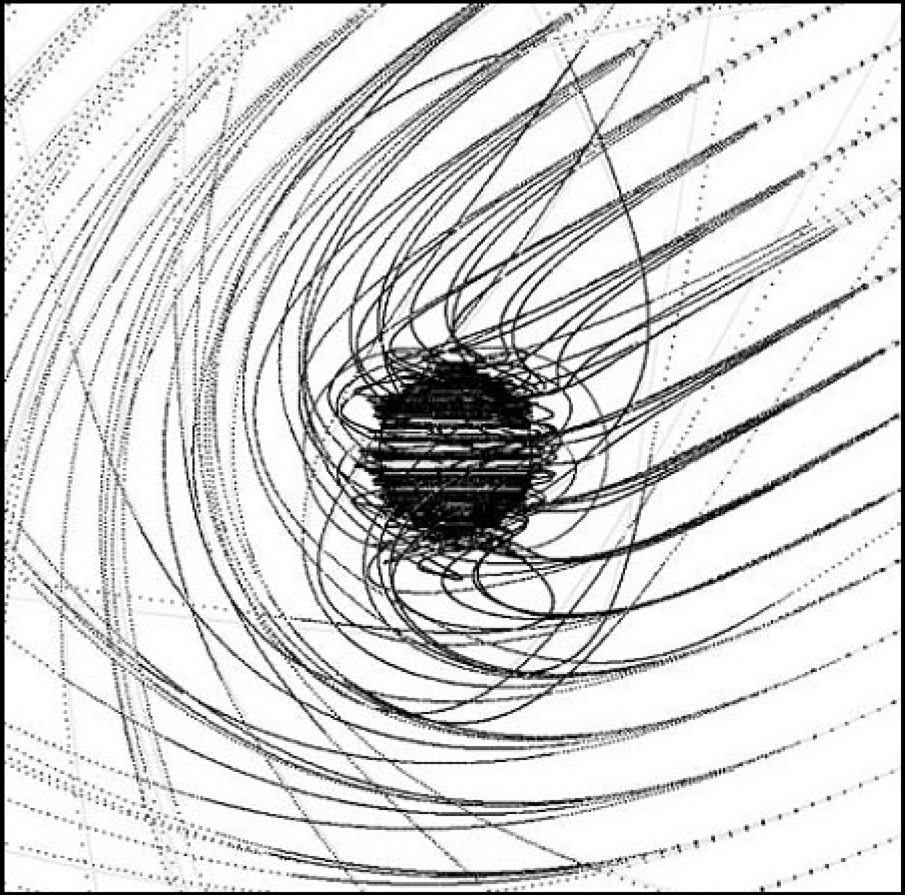
Geodésicas en el e-t de Kerr



$a = 0$, $\theta = \pi/2$, fotones



$a \neq 0$, $\theta = \pi/2$



$a \neq 0, \theta \neq \pi/2$

La métrica de Kerr tiene simetria axial → sólo p_ϕ (proyección del momento angular en la dirección del eje de rotación) es conservado.
→ el movimiento no está confinado a un plano.

p_t es conservado (indep. de t)

$$\theta = \pi/2$$

$$ds^2 = c^2\left(1 - \frac{2\mu}{r}\right) dt^2 + \frac{4\mu ac}{r}\, dt\, d\phi - \frac{r^2}{\Delta}\, dr^2 - \left(r^2 + a^2 + \frac{2\mu a^2}{r}\right) d\phi^2,$$

$$p_t = g_{tt}\dot{t} + g_{t\phi}\dot{\phi} = c^2\left(1 - \frac{2\mu}{r}\right)\dot{t} + \frac{2\mu ac}{r}\dot{\phi} = kc^2,$$

$$p_\phi = g_{\phi t}\dot{t} + g_{\phi\phi}\dot{\phi} = \frac{2\mu ac}{r}\dot{t} - \left(r^2 + a^2 + \frac{2\mu a^2}{r}\right)\dot{\phi} = -h,$$

$$\dot{t} = \frac{1}{\Delta}\left[\left(r^2 + a^2 + \frac{2\mu a^2}{r}\right)k - \frac{2\mu a}{cr}h\right],$$
$$\dot{\phi} = \frac{1}{\Delta}\left[\frac{2\mu ac}{r}k + \left(1 - \frac{2\mu}{r}\right)h\right].$$

$$g^{tt}(p_t)^2 + 2g^{t\phi}p_t p_\phi + g^{\phi\phi}(p_\phi)^2 + g^{rr}(p_r)^2 = \epsilon^2,$$

$$p_r = g_{rr}\dot{r}$$

$$\dot{r}^2 = c^2k^2 - \epsilon^2 + \frac{2\epsilon^2\mu}{r} + \frac{a^2(c^2k^2 - \epsilon^2) - h^2}{r^2} + \frac{2\mu(h - ack)^2}{r^3}.$$

$$\dot{t} = \frac{1}{\Delta}\left[\left(r^2 + a^2 + \frac{2\mu a^2}{r}\right)k - \frac{2\mu a}{cr}h\right],$$

$$\dot{\phi} = \frac{1}{\Delta}\left[\frac{2\mu ac}{r}k + \left(1 - \frac{2\mu}{r}\right)h\right].$$

A diferencia de lo que pasa em Schwarszchild, tanto *t* cuanto *Φ* son mal comportadas cerca del horizonte.

Partículas com masa no nula $\epsilon^2 = c^2$

$$\dot{r}^2 = c^2(k^2 - 1) + \frac{2\mu c^2}{r} + \frac{a^2c^2(k^2-1) - h^2}{r^2} + \frac{2\mu(h - ack)^2}{r^3}.$$

$$\tfrac{1}{2}\dot{r}^2 + V_{\text{eff}}(r; h, k) = \tfrac{1}{2}c^2(k^2 - 1),$$

$$V_{\text{eff}}(r; h, k) = -\frac{\mu c^2}{r} + \frac{h^2 - a^2c^2(k^2-1)}{2r^2} - \frac{\mu(h - ack)^2}{r^3}.$$

$$\varepsilon^2 = 0,$$

$$\dot{r}^2 = c^2k^2 + \frac{a^2c^2k^2 - h^2}{r^2} + \frac{2\mu(h-ack)^2}{r^3}.$$

$$\frac{\dot{r}^2}{h^2} + V_{\text{eff}}(r; b) = \frac{1}{b^2},$$

$$V_{\text{eff}}(r; b) = \frac{1}{r^2}\left[1 - \left(\frac{a}{b}\right)^2 - \frac{2\mu}{r}\left(1 - \frac{a}{b}\right)^2\right].$$

Rutinas de simplificación em el GRTensot

En general es conveniente simplificar el resultado antes de mostrarlo

gralter(Tensor, f1, f2, ...)

Onde *f1, f2...* son rutinas de simplificación:

Ej:

```
> gralter ( R(dn,dn), expand, factor ):
```

(1) *simplify*
(2) *trig*
(3) *power*
(4) *hypergeom*
(5) *radical*
(6) *expand*
(7) *factor*
(8) *normal*
(9) *sort*
(10) *sqrt*
(11) *trigsin*
(12) *cons*
(13) *consr*
(14) *other*

Para hacer substituciones em todas las componentes de un tensor,

```
> grmap ( R(dn,dn), subs, Q=0, 'x' ):
```

```
> grmap ( R(dn,dn), Ricciscalar, subs, b(r)=B, c(r)=C, 'x' ):
```

En casos complicados, es conveniente por ejemplo simplificar los Christoffels antes de calcular el tensor de Riemann

grcalcalter (*objectSeq*, [*function1*], [*function2*], ...)

```
> grcalcalter ( R(dn,dn), expand, factor ):
```

Reglas de simplificación

* **Casi todos los cálculos em GRTensor precisam de alguna simplificación. Lo mejor es comenzar con *normal*.**

* **Simplificar la métrica al comienzo del cálculo, y los Christoffels antes de calcular la curvatura.**

* **Usar substituciones tipo *u(θ)=cos(θ)* aceleran el cálculo.**

* **Regla general: los cálculos em GRTensor terminan rápido (algunos minutos) o precisan de simplificaciones.**

$$C^{\lambda\mu\nu\kappa}C_{\lambda\mu\nu\kappa} \qquad \frac{\varepsilon^{\lambda\mu}{}_{\rho\sigma}C^{\rho\sigma\nu\kappa}C_{\lambda\mu\nu\kappa}}{\sqrt{g}}$$

$$C_{\lambda\mu\nu\kappa}C^{\nu\kappa\rho\sigma}C_{\rho\sigma}{}^{\lambda\mu} \qquad \frac{C_{\lambda\mu\nu\kappa}C^{\nu\kappa\rho\sigma}\varepsilon_{\rho\sigma}{}^{\tau\xi}C_{\tau\xi}{}^{\lambda\mu}}{\sqrt{g}}$$

Invariantes de Weyl