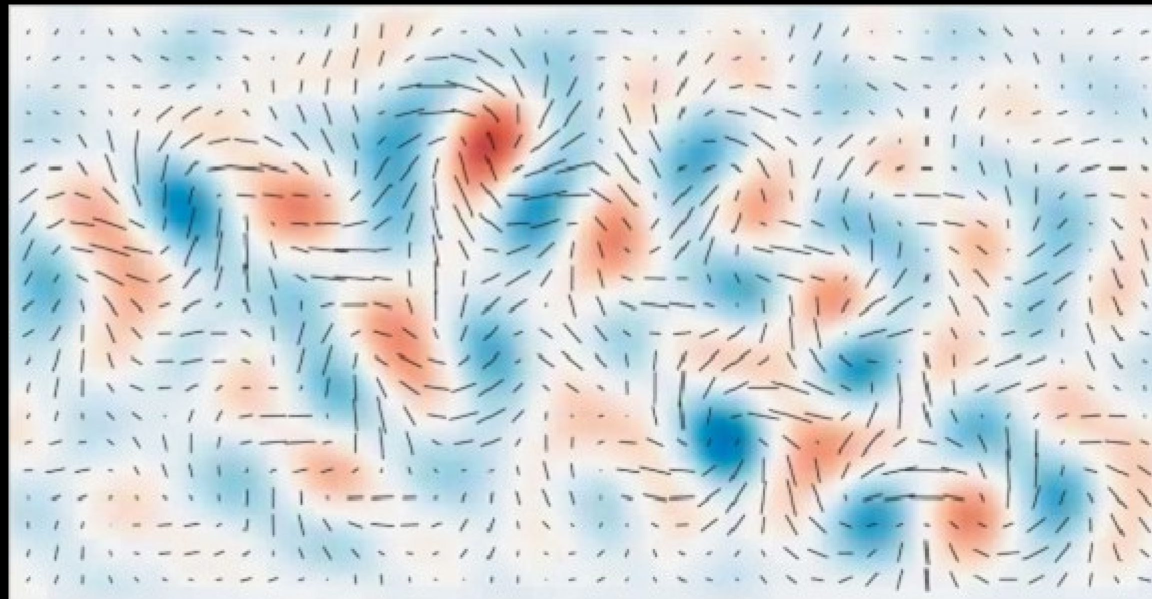

Aplicaciones de GRTensor en Astrofísica y Cosmologia

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Para entender la estructura de un dado e-t cosmológico, podemos estudiar como el campo gravitacional afecta a una familia de geodésicas (que pueden representar por ejemplo el movimiento de fotones o partículas de fluido).

Con la 4-velocidad comovil (que se toma como aquella que anula al dipolo del CMB) definida por

$$u^a = \frac{dx^a}{d\tau}, \qquad u_a u^a = -1,$$

podemos definir los proyectores

$$U^a{}_b = -u^a u_b \Rightarrow U^a{}_c U^c{}_b = U^a{}_b,\, U^a{}_a = 1,\, U_{ab}u^b = u_a, \quad \text{paralelo a } \mathbf{u}$$

$$h_{ab} = g_{ab} + u_a u_b \Rightarrow h^a{}_c h^c{}_b = h^a{}_b,\, h^a{}_a = 3,\, h_{ab}u^b = 0. \quad \text{perpendicular a } \mathbf{u}$$

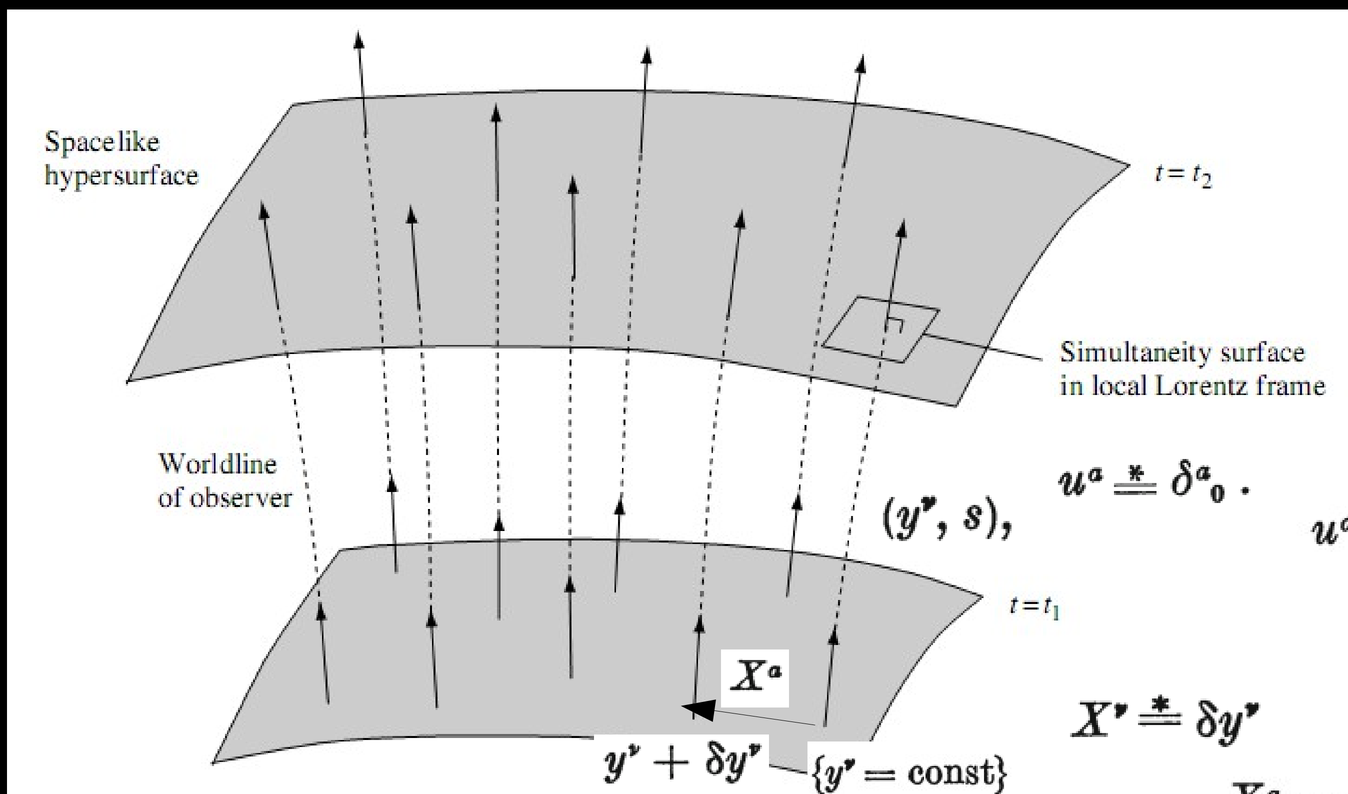
$$ds^2 := g_{ab}\, dx^a\, dx^b = h_{ab}\, dx^a\, dx^b - (u_a\, dx^a)^2;$$

Usaremos dos derivadas:

$$\dot{T}^{ab}{}_{cd} = u^e \nabla_e T^{ab}{}_{cd}\,, \qquad \tilde{\nabla}_e T^{ab}{}_{cd} = h^a{}_f h^b{}_g h^p{}_c h^q{}_d h^r{}_e \nabla_r T^{fg}{}_{pq}\,,$$

Notación: $v^{\langle a\rangle} = h^a{}_b v^b, \qquad T^{\langle ab\rangle} = [h^{(a}{}_c h^{b)}{}_d - \frac{1}{3}h^{ab}h_{cd}]T^{cd};$

Consideremos dos geodésicas vecinas em el sistema comóvil:



$$u^a = \left.\frac{\mathrm{d}x^a}{\mathrm{d}s}\right|_{y^\nu=\text{const}}$$

$$X^a = (\partial x^a/\partial y^\nu)\,\delta y^\nu.$$

$$\dot{X}^a = X^a{}_{;b}u^b = \{\partial(\partial x^a/\partial y^\nu\,\delta y^\nu)/\partial x^b + \Gamma^a{}_{bc}(\partial x^c/\partial y^\nu\,\delta y^\nu)\}\frac{\partial x^b}{\partial s} =$$

$$= \{\partial(\partial x^a/\partial s)/\partial x^c + \Gamma^a{}_{cb}\,\partial x^b/\partial s\}\frac{\partial x^c}{\partial y^\nu}\delta y^\nu = u^a{}_{;b}X^b\,,$$

$$\partial^2 x^a/\partial y^\nu\partial s = \partial^2 x^a/\partial s\,\partial y^\nu \quad \text{and} \quad \Gamma^a{}_{bc} = \Gamma^a{}_{cb}\,.$$

$$\dot{X}_a = u_{a;b}X^b\,.$$

Vector posición relativa $X_{\perp a} := h_a{}^b X_b\,.$

Vector velocidad relativa $V^a := h^a{}_b(X_\perp{}^b)^\cdot\,,$

$$V^a = h^a{}_b(X^c h_c{}^b)_{;d}\,u^d\,.$$

$$V^a = v^a{}_b X_\perp{}^b\,,$$

$$v_{ab} := h_a{}^c h_b{}^d u_{c;d}\,;$$

$$\nabla_a u_b = -u_a \dot{u}_b + \tilde{\nabla}_a u_b$$

v_{ab}

$$\dot{u}^a = u^b \nabla_b u^a$$

aceleración (interacciones diferentes de la gravitación)

$$\tilde{\nabla}_a u_b = \theta_{ab} + \omega_{ab}$$

$$\theta_{ab} = \theta_{(ab)}\,, \qquad \omega_{ab} = \omega_{[ab]}\,,$$

$$\theta_{ab} u^b = 0 = \omega_{ab} u^b\,,$$

$$\theta_{ab} = \sigma_{ab} + \tfrac{1}{3}\theta h_{ab}\,,$$

$$\sigma^a{}_a = 0\,,$$

$$\sigma_{ab} u^b = 0$$

$$\sigma_{ab} = \sigma_{(ab)}\,,$$

$$\nabla_a u_{\rm b} = -u_a \dot{u}_{\rm b} + \tilde{\nabla}_a u_{\rm b} = -u_a \dot{u}_{\rm b} + \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab}$$

$$\Theta = \tilde{\nabla}_a u^a$$

$$\sigma_{ab} = \tilde{\nabla}_{\langle a} u_{b\rangle}$$

$$\sigma_{ab} = \sigma_{(ab)}, \ \sigma_{ab} u^b = 0, \ \sigma^a{}_a = 0$$

$$\omega_{ab} = \tilde{\nabla}_{[a} u_{b]}$$

$$\omega_{ab} = \omega_{[ab]}, \ \omega_{ab} u^b = 0$$

$$X_{\perp}{}^{a} = n^{a}\,\delta l \qquad n_{a} n^{a} = 1, \qquad (\delta l)^2 = h_{ab} X^a X^b$$

$$V^a := h^a{}_b (X_{\perp}{}^b)^{\bullet}\,, \qquad V^a = v^a{}_b X_{\perp}{}^b\,, \qquad v_{ab} := h_a{}^c h_b{}^d u_{c;d}\,;$$

$$\frac{(\delta l)^{\bullet}}{(\delta l)} = \theta_{ab} n^a n^b = \sigma_{ab} n^a n^b + \frac{1}{3}\theta\,, \qquad h_a{}^b (n_b)^{\bullet} = (\omega_a{}^b + \sigma_a{}^b - (\sigma_{cd} n^c n^d)\, h_a{}^b)\, n_b\,.$$

Cantidades cinemáticas

$\Theta = \tilde{\nabla}_a u^a$ describe cambios en el volumen del fluido

$\sigma_{ab} = \tilde{\nabla}_{\langle a} u_{b\rangle}$ describe la distorsión del fluido

$\omega_{ab} = \tilde{\nabla}_{[a} u_{b]}$ describe la rotación del fluido (com relación a un sist. de ref. sin rotación que se propaga paralelamente)

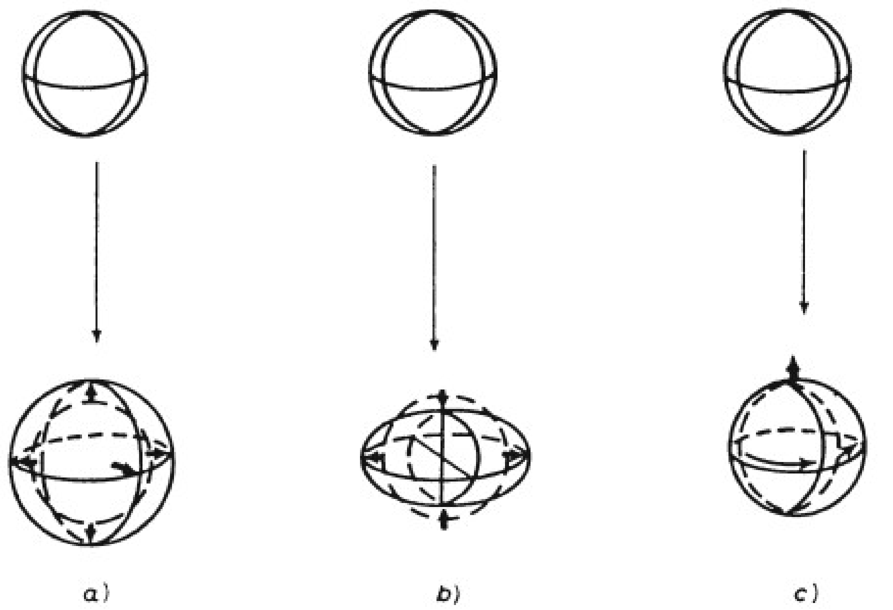


Fig. 1. – During a small time interval, *a*) the action of θ alone transforms a fluid sphere to a similar sphere of different volume but with the same orientation. *b*) The action of σ_{ab} alone distorts the sphere, leaving its volume constant and the directions of the principal axes of shear unchanged. *c*) The action of ω_a alone is to give a rigid rotation leaving one direction (the axis of rotation) fixed. As time progresses the directions of the principal axes of shear and of the axis of rotation will, in general, change.

$$\omega^a = \frac{1}{2}\eta^{abc}\omega_{bc} \Rightarrow \omega_a u^a = 0, \omega_{ab}\omega^b = 0,$$

Vector vorticidad

$$\eta_{abc} = u^d \eta_{dabc} \Rightarrow \eta_{abc} = \eta_{[abc]}, \eta_{abc}u^c = 0,$$

$$\eta_{abcd} = \eta_{[abcd]}, \eta_{0123} = \sqrt{|\det g_{ab}|})$$

$$\omega^2 = \frac{1}{2}(\omega_{ab}\omega^{ab}) \geq 0, \sigma^2 = \frac{1}{2}(\sigma_{ab}\sigma^{ab}) \geq 0,$$

En el caso de la métrica de FLRW,

$$\dot{u} = 0 = \omega = \sigma = 0,$$

$$E_{ab} = H_{ab} = 0.$$

CantcinFLRW.mws

La ec de Raychaudhuri

$$u_{a;d;c} - u_{a;c;d} = R_{cbcd} u^b .$$

Multiplicando por u^d :

$$(u_{a;c})^{\bullet} - \dot{u}_{a;c} + u_{a;d} u^d{}_{;c} + R_{abcd} u^b u^d = 0 .$$

Proyectando y usando la definición de v_{ab} :

$$h_a{}^c h_b{}^d (v_{cd})^{\bullet} - \dot{u}_a \dot{u}_b - h_a{}^c h_b{}^d \dot{u}_{c;d} + v_{ad} v^d{}_b + R_{acbd} u^c u^d = 0 .$$

Tomando la traza en *a* y *b* (y considerando aceleración cero):

$$\xi^c \nabla_c \theta = \frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{cd}\xi^c\xi^d$$

$$\xi^c \nabla_c \theta = \frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{cd}\xi^c\xi^d$$

Usando las ecs de Einstein :

$$\dot{\Theta} - \tilde{\nabla}_a \dot{u}^a = -\tfrac{1}{3}\Theta^2 + (\dot{u}_a \dot{u}^a) - 2\sigma^2 + 2\omega^2 - \tfrac{1}{2}(\mu + 3p) + \Lambda\,,$$

En el caso de la sol. de FLRW:

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leqq 0 \quad ,$$

$$\frac{d}{d\tau}(\theta^{-1}) \geqq \frac{1}{3}$$

$$\theta^{-1}(\tau) \geqq \theta_0^{-1} + \frac{1}{3}\tau$$

Si θ0 < 0 , θ → ∞ para algun τ → singularidad.

LEMMA 9.2.1. Let ξ^a be the tangent field of a hypersurface orthogonal timelike geodesic congruence. Suppose $R_{ab}\xi^a\xi^b \geqq 0$, as will be the case if Einstein's equation holds in the spacetime and the strong energy condition is satisfied by the matter. If the expansion θ takes the negative value θ_0 at any point on a geodesic in the congruence, then θ goes to $-\infty$ along that geodesic within proper time $\tau \leqq 3/|\theta_0|$.

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Definiendo objetos em el GRTensor - El comando grdef()

Dos formas del grdef:

```
> grdef ( `A{a b}` ):                    A(dn,dn)

> grdef ( `G2{a b} := R{a b} - (1/2)*Ricciscalar*g{a b} + lambda*g{a b}` ):
```

grdef crea el tensor pero no calcula sus componentes → grcalc

grdef (*defString*, [*symSet*], [*asymSet*], [*rsumSet*])

índices

```
          a, b, c, ...
       ^a, ^b, ^c, ...
    (a), (b), (c), ...
^(a), ^(b), ^(c), ...
```

Los nombres de los indices em la definicion no pueden tener valores asignados en cualquier otra parte de la hoja.

Derivadas:

`R{^a b ,c}:`	$R^{a}{}_{b,c} := \frac{\partial R_{ab}}{\partial x^{c}}$
`R{^a b ;c}:`	$R^{a}{}_{b;c} := R^{a}{}_{b,c} - \Gamma^{d}_{bc} R^{a}{}_{d} + \Gamma^{a}_{bd} R^{d}{}_{c}$
`R{^(a) (b) ,(c)}:`	$R^{(a)}{}_{(b),(c)} := \frac{\partial R^{(a)}{}_{(b)}}{\partial x^{d}} e_{(c)}{}^{d}$
`R{^(a) (b) ;(c)}:`	$R^{(a)}{}_{(b);(c)} := R^{(a)}{}_{(b),(c)} - \gamma^{(d)}{}_{(b)(c)} R^{(a)}{}_{(d)} + \gamma^{(a)}{}_{(b)(d)} R^{(d)}{}_{(c)}$

Algebra:

$$\frac{1}{2}\left(R_{abcd} + f(x) g_{ab} g_{cd}\right)$$

```
(1/2) * ( R{a b c d} + f(x)*g{a b}*g{c d} )
```

Contracción:

$$\sum_b R^{ab}{}_{bc}$$ → `R{^a ^b b c},`

$$\sum_b R_a{}^b R_{bc}$$ → `R{a ^b}*R{b c}.`

Operadores:

```
> grdef ( `X := R{^a ^b}*Box[ R{a b} ]` ):
```

$$X := R^{ab}\Box R_{ab},$$

$$T_{ab} := \Box(R_{acdb}R^{cd}),$$

```
> grdef ( `T1{a b} := R{a c d b}*R{^c ^d}` ):
> grdef ( `T{a b} := Box[T1{a b}]` ):
```

Simetrización

$$T_{a(bcd)} := \frac{1}{6}\left(T_{abcd} + T_{acdb} + T_{adbc} + T_{abdc} + T_{adcb} + T_{bdc}\right)$$

```
T{a (b c d)},
```

Antisimetrización

$$T_{a[bcd]} := \frac{1}{6}\left(T_{abcd} + T_{acdb} + T_{adbc} - T_{abdc} - T_{adcb} - T_{bdc}\right)$$

```
T{ a [b c d]}.
```

$R^a{}_{bc(d}R_{e)f}$

```
R{^a b c (d}*R{e) f}
```

```
T{(a ^b c)}
```

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$$T_{a(b|c|d)} := \frac{1}{2}\left(T_{abcd} + T_{adcb}\right),$$

```
T{a (b | c | d)}.
```

Simetrías

```
> grdef ( `A{(a b)(c d)}` ):
```

$$A_{abcd} = A_{bacd} = A_{abdc} = A_{badc}.$$

```
> grdef ( `A{[a b c] d}` ):
```

$$A_{abcd} = A_{bcad} = A_{cabd} = -A_{acbd} = -A_{bacd} = -A_{cbad}.$$

```
> grdef ( `A{(a b)} := B{(a b)}` ):
```

$$A_{ab} := \frac{1}{2}\left(B_{ab} + B_{ba}\right)$$

```
> grdef ( `A{a b c d}`, sym={[1,2],[3,4]} ):
```

$$A_{abcd} = A_{bacd} = A_{abdc} = A_{badc}.$$

```
> grdef ( `A{a b c d}`, sym={[2,3,4]} ):
```

$$A_{abcd} = A_{acdb} = A_{adbc} = A_{abdc} = A_{acbd} = A_{adcb}.$$

```
> grdef ( `A{a b c}`, asym={[1,2]} ):
```

$$A_{abc} = -A_{bac},$$

El comando no permite crear simetrias em grupos de indices, tal como

$$R_{abcd} = R_{cdab}.$$

Pero podemos definir tensores com la misma simetria de uno ya definido:

```
> grdef ( `T{a b c d}`, symfn=R(dn,dn,dn,dn) ):
```

Vectores: como hasta ahora o

```
> grdef ( `v^a := [0,0,0,1]` ):
```

```
> grdef ( `v{^a} := f(t)*kdelta{^a $t}` ):
```

```
> grdef ( `v{^a} := [0,0,0,f(t)]` ):
```

$$T_{ab} := P(r,t)g_{ab} + (P(r,t) + \rho(r,t))\delta^t{}_a\delta^t{}_b,$$

```
> grdef ( `T{(a b)} :=
        P(r,t)*g{a b} + ( P(r,t) + rho(r,t) )*kdelta{a $t}*kdelta{b $t}` ):
```

Los objetos definidos pueden ser grabados:

`grsavedef` (*objectSeq* , *fileName*)

```
> grsavedef ( G2(dn,dn), T(dn,dn,dn,dn), ‘newdefs.mpl‘ ):
```

Cuidado: si el archivo ya existe, escribe el nuevo y borra el anterior.

`grloaddef` (*fileName*)