Scientific Philosophy

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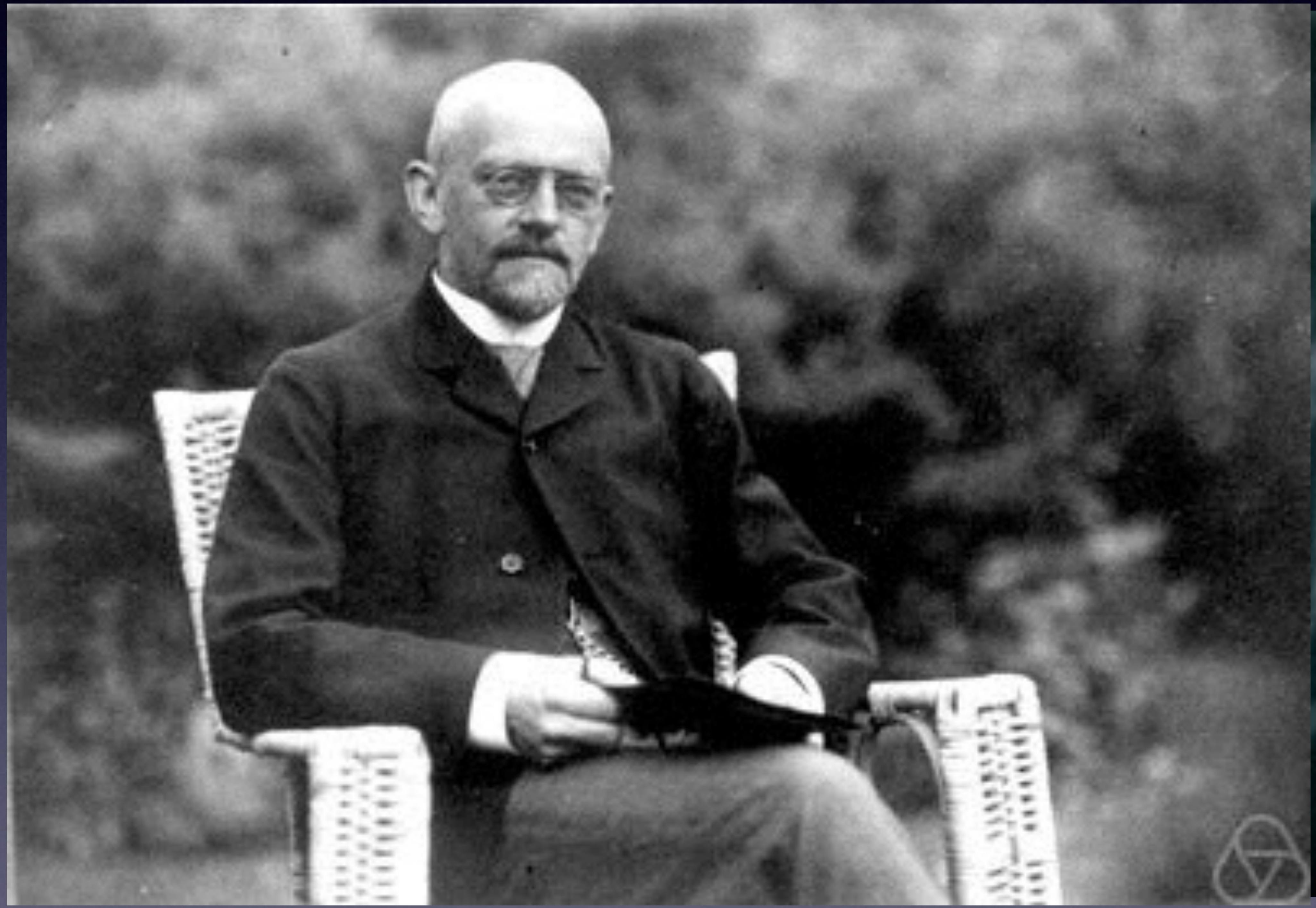
Philosophy of mathematics



The philosophy of mathematics is the branch of philosophy that studies the philosophical assumptions, foundations, and implications of mathematics.

Traditional philosophical problems in mathematics are:

- What is the ontological status of mathematical entities?
- What does it mean to refer to a mathematical object?
- What is the character of a mathematical proposition?
- What is the relation between logic and mathematics?
- What are the objectives of mathematical inquiry?
- What is the relation of mathematics with experience?
- What is mathematical beauty?
- What is the source and nature of mathematical truth?
- What is the relationship between the abstract world of mathematics and the material universe?

Platonism

Platonism is realism regarding to mathematical objects such as numbers, functions, and sets.

According Platonism, mathematics are not invented, but ***discovered***. For platonist mathematical entities are abstract in the sense that they have no spatiotemporal or causal properties, and are eternal and unchanging.

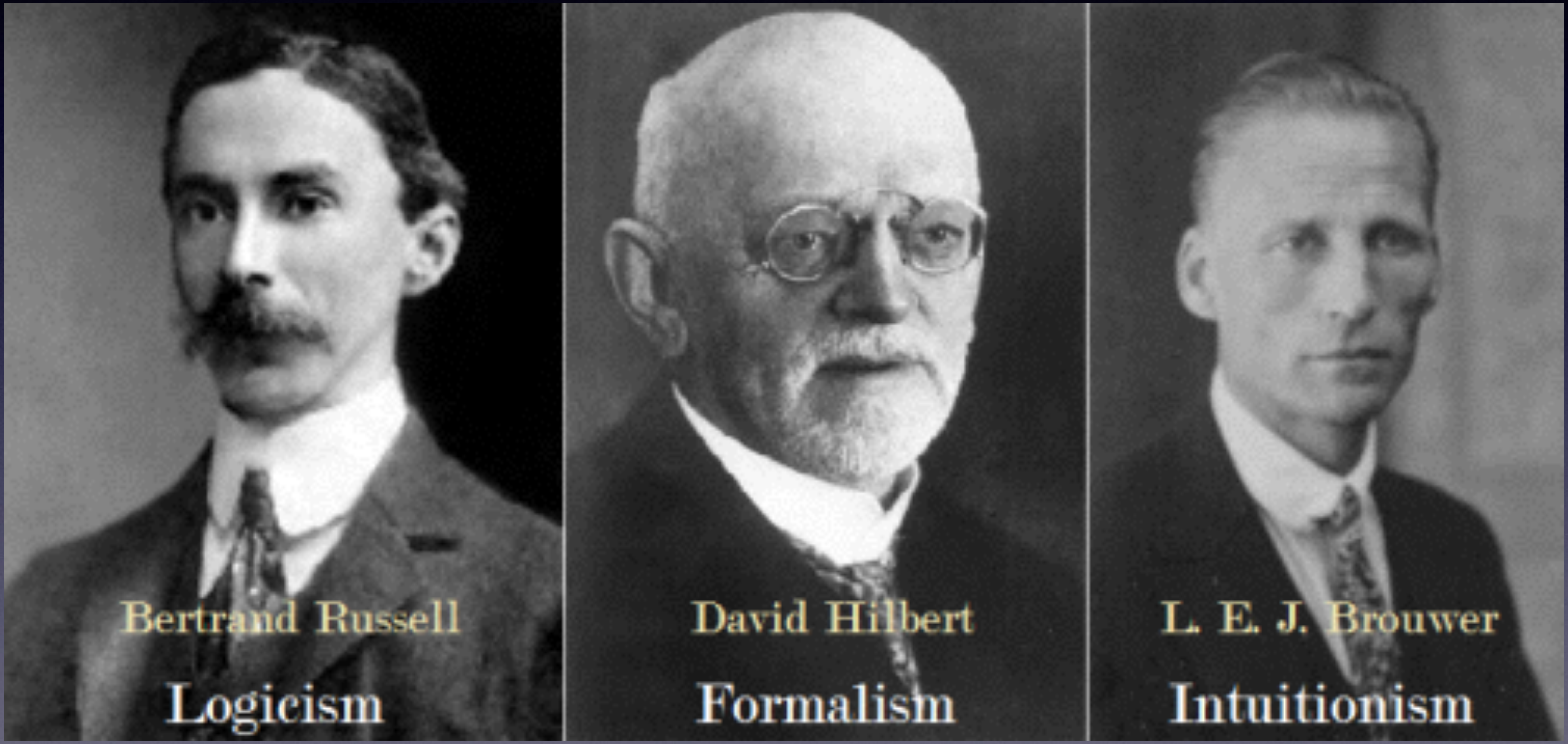
A major problem for Platonism is how we acquire knowledge of such an abstract realm and why empirical sciences that use mathematics are so successful.

Other forms of ***mathematical realism*** includes ***empiricism*** (Quine and Putnam) and ***mathematical monism*** (Tegmark).

According to the former mathematical facts are found by empirical research, just like facts in any of the other sciences. If science requires, say, numbers to explain the world, then numbers should exist.

According to monism, only mathematical objects exist. Tegmark's sole postulate is: *All structures that exist mathematically also exist physically.* <u>I do not profess to understand this claim.</u>

Major schools in the philosophy of mathematics



Logicism

Logicism is the thesis that mathematics is reducible to logic, and hence it is a part of logic.

Logicists hold that mathematics can be known a priori, but suggest that our knowledge of mathematics is just part of our knowledge of logic in general, and is thus analytic, not requiring any special faculty of mathematical intuition. In this view, logic is the proper foundation of mathematics, and all mathematical statements are necessary logical truths.

Logicism

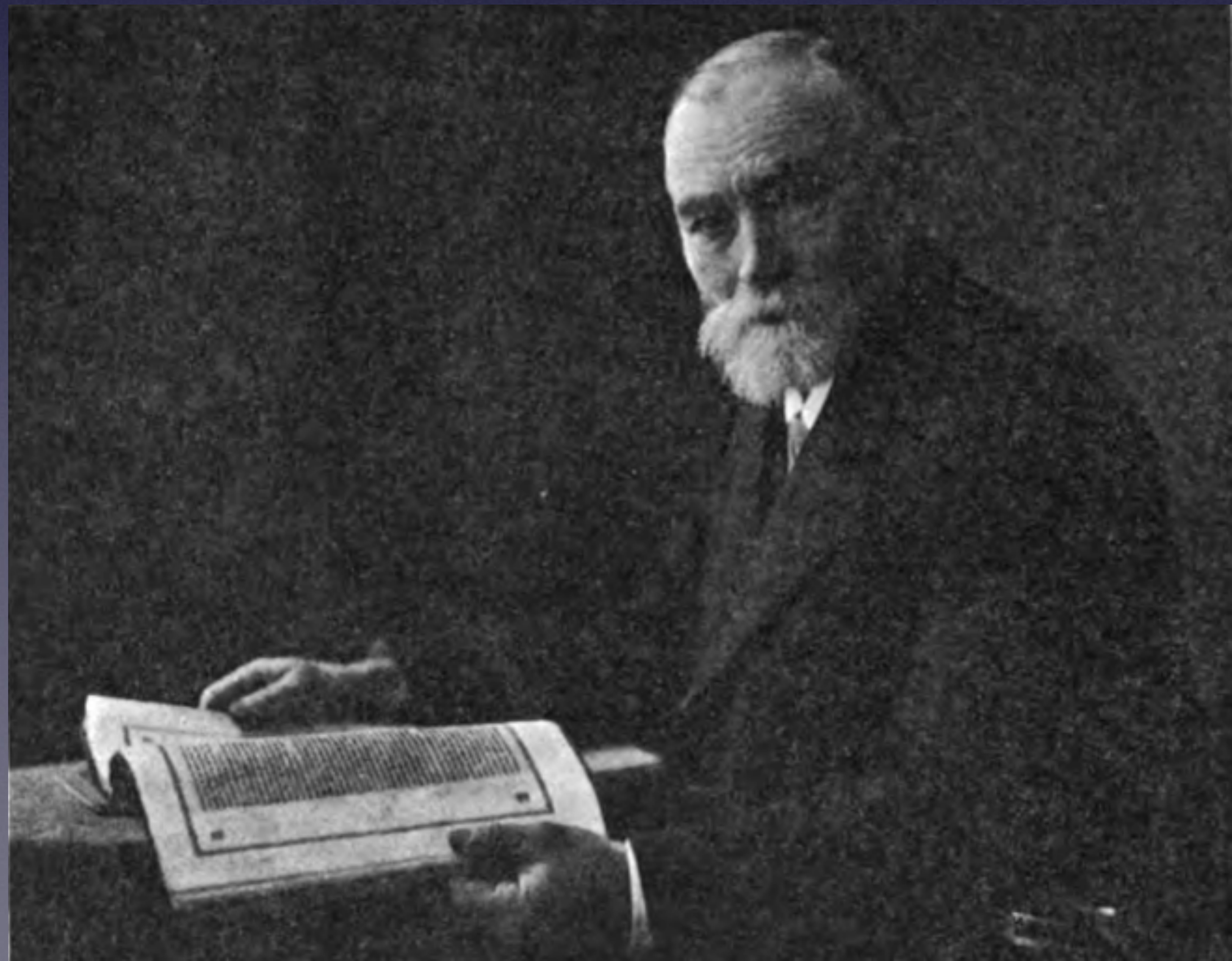
Rudolf Carnap (1931) presents the logicist thesis in two parts:

1. The concepts of mathematics can be derived from logical concepts through explicit definitions.
2. The theorems of mathematics can be derived from logical axioms through purely logical deduction.

Carnap, Rudolf (1931), "Die logizistische Grundlegung der Mathematik", Erkenntnis 2, 91-121.

Logicism: historical remarks.

The idea that mathematics is logic in disguise goes back to Leibniz. But a serious attempt to carry out the logicist program in detail could be made only when in the nineteenth century the basic principles of central mathematical theories were articulated (by Dedekind and Peano) and the principles of logic were uncovered (by Frege).



Gottlob Frege

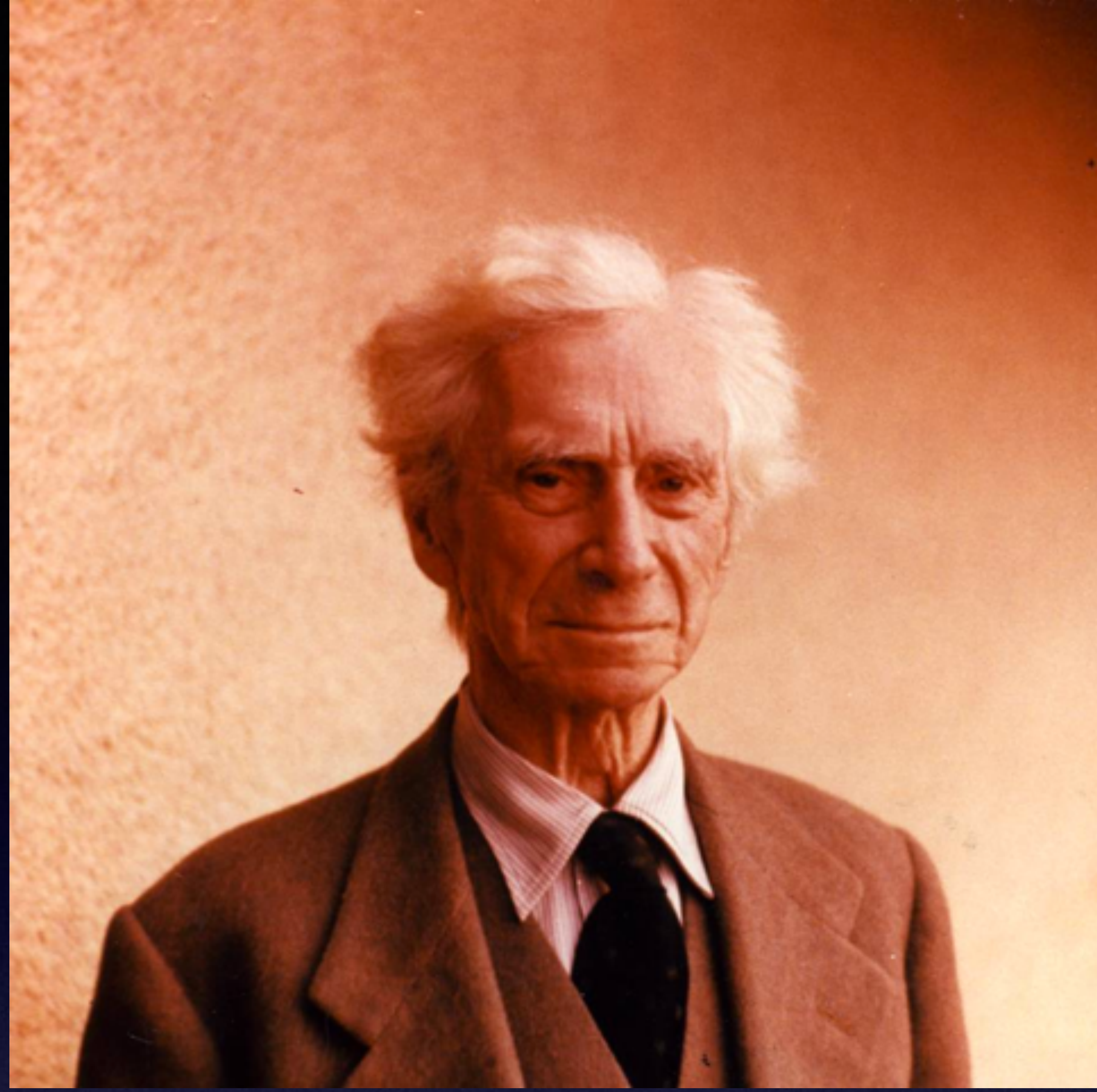
Logicism: historical remarks.

Frege devoted much of his career to trying to show how mathematics can be reduced to logic. He managed to derive the principles of Peano arithmetic from the basic laws of a system of second-order logic. His derivation was flawless. However, he relied on one principle which turned out not to be a logical principle after all. Even worse, it is untenable. The principle in question is Frege's Basic Law V:

$$\{x|Fx\}=\{x|Gx\} \equiv \forall x(Fx \equiv Gx),$$

In words: the set of the Fs is identical with the set of the Gs iff the Fs are precisely the Gs. In a famous letter to Frege, Russell showed that Frege's Basic Law V entails a contradiction (Russell 1902).

Russell’s paradox



Let us consider the class of all classes that are not members of themselves. Let us call this class A. Then if

$$A \in A \to A \notin A \text{ and if } A \notin A \to A \in A.$$

Problems with logicism

1. Logic is semantically neutral, but mathematics is interpreted. It is not possible to derive semantics from syntax.
2. In logicist constructions of mathematical theories non-logical concepts such as “sequence” appear.
3. Gödel incompleteness theorem seems to pose insurmountable obstacles to mathematical construction based entirely on logic.

Gödel Theorems

The incompleteness theorems apply to formal systems that are of sufficient complexity to express the basic arithmetic of the natural numbers and which are consistent, and effectively axiomatized. The incompleteness theorems are about formal provability within these systems. There are several properties that a formal system may have, including completeness, consistency, and the existence of an effective axiomatization. The incompleteness theorems show that systems which contain a sufficient amount of arithmetic cannot possess all three of these properties.

Gödel Theorems

First Incompleteness Theorem: "Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e., there are statements of the language of F which can neither be proved nor disproved in F."

Gödel Theorems

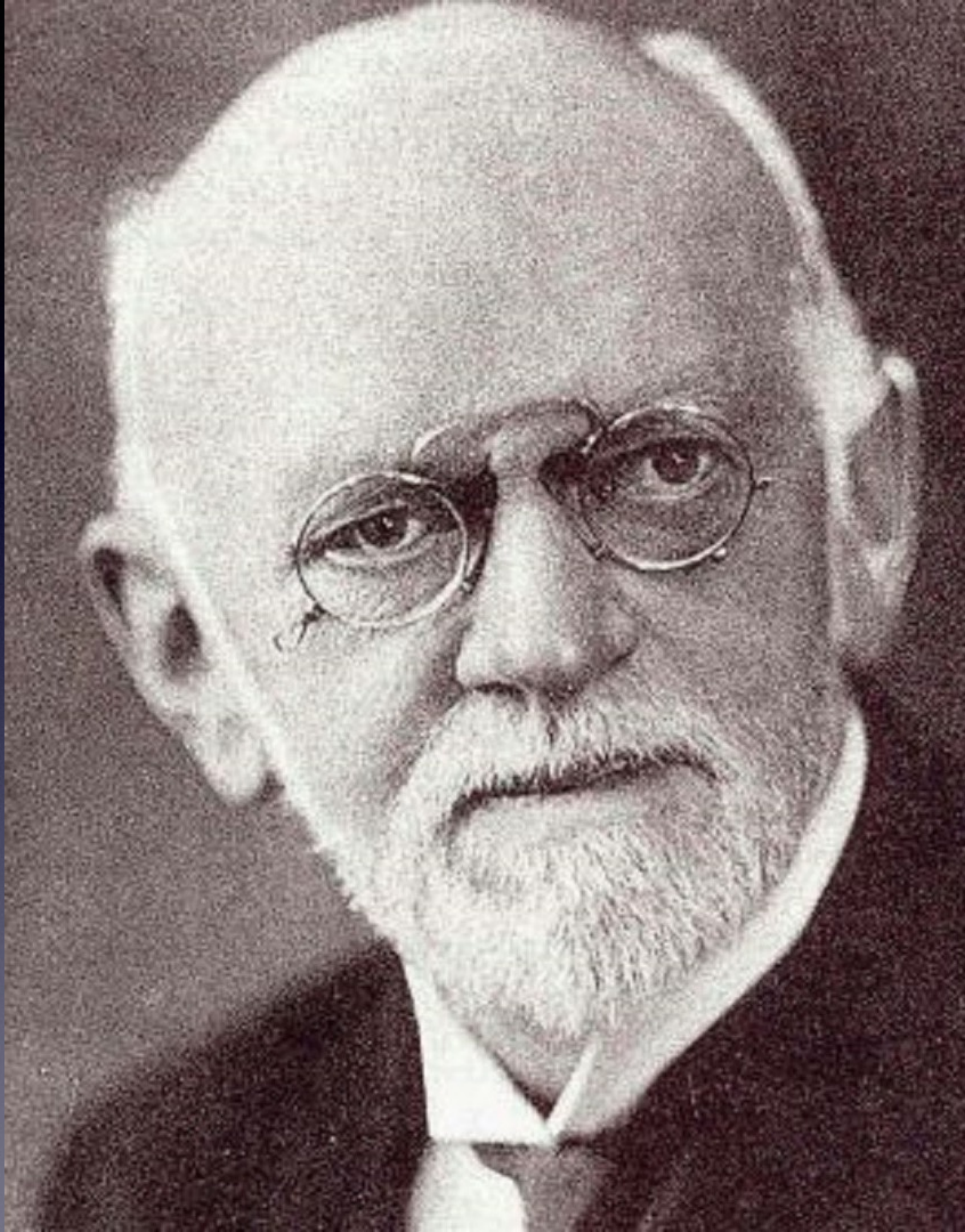
Second Incompleteness Theorem: "Assume F is a consistent formalized system which contains elementary arithmetic. Then $F \nvdash \mathrm{Cons}(F)$ "

Formalism

Formalism holds that statements of mathematics and logic can be considered to be statements about the consequences of certain string manipulation rules.

According to formalism, the statements expressed in logic and mathematics are not about numbers, sets, or triangles or any other subject matter — in fact, they aren't "about" anything at all. They are syntactic forms whose shapes and locations have no meaning unless they are given an interpretation (or semantics).

A major early proponent of formalism was **David Hilbert**, whose program was intended to be a complete and consistent axiomatization of all of mathematics. Hilbert aimed to show the consistency of mathematical systems from the assumption that the "finitary arithmetic" (a subsystem of the usual arithmetic of the positive integers, chosen to be philosophically uncontroversial) was consistent.



Hilbert's goals of creating a system of mathematics that is both complete and consistent were seriously undermined by the second of **Gödel**'s incompleteness theorems, which state that sufficiently expressive consistent axiom systems can never prove their own consistency. Since any such axiom system would contain the finitary arithmetic as a subsystem, Gödel's theorem implied that it would be impossible to prove the system's consistency relative to that (since it would then prove its own consistency, which Gödel had shown was impossible).

A revised version of formalism is known as ***deductivism***. In deductivism, one assigns meaning to the strings in such a way that the rules of the game become true (i.e., true statements are assigned to the axioms and the rules of inference are truth-preserving). Then one must accept the theorem, or, rather, the interpretation one has given it must be a true statement. Thus, formalism needs not mean that mathematics is nothing more than a meaningless symbolic game. It is usually hoped that there exists some interpretation in which the rules of the game hold.

Objections to formalism

The main critique of formalism is that the actual mathematical ideas that occupy mathematicians are far removed from the string manipulation games mentioned above. Formalism is thus silent on the question of which axiom systems ought to be studied, as none is more meaningful than another from a formalistic point of view.

Many formalists would say that in practice, the axiom systems to be studied will be suggested by the demands of science or other areas of mathematics.

Intuitionism

Intuitionism involves the regulative principle that only mathematical entities which can be explicitly constructed in a certain sense should be admitted to mathematical discourse. In this view, mathematics is an exercise of the human intuition, not a game played with meaningless symbols. Instead, it is about entities that we can create directly through mental activity. In addition, some adherents of these schools reject non-constructive proofs, such as a proof by contradiction.

A major force behind intuitionism was **Luitzen E.J. Brouwer**, who rejected the usefulness of formalised logic of any sort for mathematics. His student Arend Heyting postulated an intuitionistic logic, different from the classical Aristotelian logic; this logic does not contain the law of the excluded middle and therefore deprecates proofs by contradiction.

Objections to intuitionism

Intuitionism must abandon important parts of mathematics that are demonstrated in non-consecutive ways. It also cannot deal with the actual infinite. In addition, the concepts of "construction" and "intuition" are not well defined.

Fictionalism

Fictionalism is a view on the nature of mathematical objects. The central point of the fictionalist strategy is to emphasise that mathematical entities are like fictional entities. They have similar features that fictional objects such as Sherlock Holmes or Hamlet have.

Fictionalism

The fictionalist's proposal is to consider mathematical objects as ***abstract artifacts***.

Fictional objects are created by the intentional acts of their authors (in this sense, they are artifacts). So, they are introduced in a particular context, in a particular time.

Fictionalism

Similarly, mathematical entities are created, in a particular context, in a particular time. They are ***artifacts.***

Mathematical entities are created when constitution principles are put forward to describe their constitution and role into a system, and when consequences are drawn from such principles.

Mathematical entities thus introduced are also dependent on (i) the existence of particular copies of the works in which such comprehension principles have been presented (or memories of these works), and (ii) the existence of a community who is able to understand these works. It's a perfectly fine way to describe the mathematics of a particular community as being lost if all the copies of their mathematical works have been lost and there's no memory of them.

Fictionalism

Thus, mathematical entities, introduced via the relevant comprehension principles, turn out to be contingent—at least in the sense that they depend on the existence of particular concrete objects in the world, such as, suitable mathematical works. ***They do not exist independently of human beings that invent them. <u>Fictionalism is a materialist theory of mathematics.</u>***

Fictionalism

The fictionalist insists that there is nothing mysterious about how we can refer to mathematical objects and have knowledge of them.

Reference to mathematical objects is made possible by the works in which the relevant comprehension principles are formulated. In these works, via the relevant principles, the corresponding mathematical objects are introduced. The principles specify the meaning of the mathematical terms that are introduced as well as the properties that the mathematical objects that are thus posited have. In this sense, the comprehension principles provide the context in which we can refer to and describe the mathematical objects in question.

Fictionalism

Our knowledge of mathematical objects is then obtained by examining the attributes these objects have, and by drawing consequences from the comprehension principles.

Ontological assumptions of mathematics

There are two types of commitment: ***quantifier commitment*** and ***ontological commitment***. We incur quantifier commitment to the objects that are in the range of our quantifiers. We incur ontological commitment when we are committed to the existence of certain objects. However, despite Quine's view, quantifier commitment doesn't entail ontological commitment. Fictional discourse and mathematical discourse illustrate that.

Ontological assumptions of mathematics

This can be made by invoking a distinction between partial quantifiers and the existence predicate.The idea is to resist reading the existential quantifier as carrying any ontological commitment. Rather, ***the existential quantifier only indicates that the objects that fall under a concept (or have certain properties) are less than the whole domain of discourse.*** To indicate that the whole domain is invoked (e.g. that every object in the domain have a certain property), we use a universal quantifier.

Ontological assumptions of mathematics

Two different functions are clumped together in the traditional, Quinean reading of the existential quantifier: (i) to assert the existence of something, on the one hand, and (ii) to indicate that not the whole domain of quantification is considered, on the other. These functions are best kept apart. ***We should use a partial quantifier (that is, an existential quantifier free of ontological commitment) to convey that only some of the objects in the domain are referred to, and introduce an existence predicate in the language in order to express existence claims.*** By distinguishing these two roles of the quantifier, we also gain expressive resources.

Ontological assumptions of mathematics

Suppose that "∃" stands for the partial quantifier and "E" stands for the existence predicate. In this case, we can express: $\exists x\,(Fx \wedge \neg Ex)$, that means "some objects have the property F and they are not real (or they do not exists)".

Consequences of fictionalism

(1) ***Mathematical knowledge***: Understanding and hence knowledge of mathematical entities, just as knowledge of fictional entities in general, is the result of producing suitable descriptions of the objects in question and drawing consequences from the assumptions that are made.

Consequences of fictionalism

(2) ***Reference to mathematical entities***: How is reference to mathematical objects accommodated in the fictionalist's approach? The adopted principles specify some of the properties that the objects that are introduced have, and by invoking these properties, it's possible to refer to the objects in question as those objects that have the corresponding properties.

Mathematical reference is always contextual: it's made in the context of the comprehension principles that give meaning to the relevant mathematical terms.

Consequences of fictionalism

(3) ***Application of mathematics***: For the fictionalist, the application of mathematics is a matter of using the expressive resources of mathematical theories to accommodate different aspects of scientific discourse. ***The only requirement is that the mathematical theory be consistent, i.e. free of contradictions. Then, in mathematics, the truth criterion is internal coherence.***

Summing up: Mathematics can be understood as the study and development of ***fictionally interpreted formal systems that are closed under deduction.*** These systems are not purely syntactic as the logistic systems. The are interpreted, but their class of reference is formed by ***conceptual artifacts***. These are human abstract constructions with exists only in the context of a certain formalism where they are introduced. Hence, ***mathematics has no ontological import***. The referents of mathematics cannot exist independently of the human mind.